Lecture 3

Plasma waves I

3.1 Aims and Learning Outcomes

The **Aim** of this lecture is to introduce and explain the characteristics, physics, and analytic derivations for the common wave modes in the collisionless plasmas characteristic of space and astrophysical systems.

Expected Learning Outcomes. You are expected to be able to

- Understand and explain why plasma waves are important.
- Identify and describe the common plasma wave modes.
- Follow the derivations for the common plasma wave modes.
- Use plasma waves to diagnose the characteristics of a plasma.

3.2 Why are plasma waves of interest?

Numerous reasons exist for why plasma waves are important and/or useful. Here waves includes both local waves that cannot travel effectively to other plasmas (e.g., electrostatic waves) and free-space electromagnetic radiation that can travel to us from a remote source. Some reasons are listed in the Course Handout, that was distributed in the Introductory Lecture, and in Lecture 2. For convenience, these and other rationales are described next, before we start exploring the characteristics of plasma waves.

- Waves identify the characteristic resonance frequencies of plasmas (such as the electron plasma frequency or the ion gyrofrequency), so they can be used as diagnostics for the physical properties of the plasma.
- The generation of enhanced, nonthermal levels of waves can be used as a diagnostic for the presence and characteristics of a source of particle free energy in the plasma. These include the presence of electron or ion beams and temperature anisotropies.
- Bursts of waves and radiation are diagnostics of energy releases in plasmas, for instance in solar flares or magnetospheric substorms.
- Waves and radiation carry energy, momentum, and information from one region/plasma to another.

- Plasma waves are often responsible for heating plasmas, the acceleration of energetic particles, limiting anisotropies that would otherwise lead to macroscopic plasma instabilities, and removing free energy. Put another way, plasma waves often allow a plasma to become unstable or bursty on a microscopic and mesoscopic level, preventing the macroscopic plasma from becoming unstable due to charge separations, pressure imbalances etc.
- Electromagnetic radiation is the primary, and usually only, diagnostic of remote plasmas. This is true for essentially all astrophysical phenomena and also for many solar system phenomena.

3.3 Overview

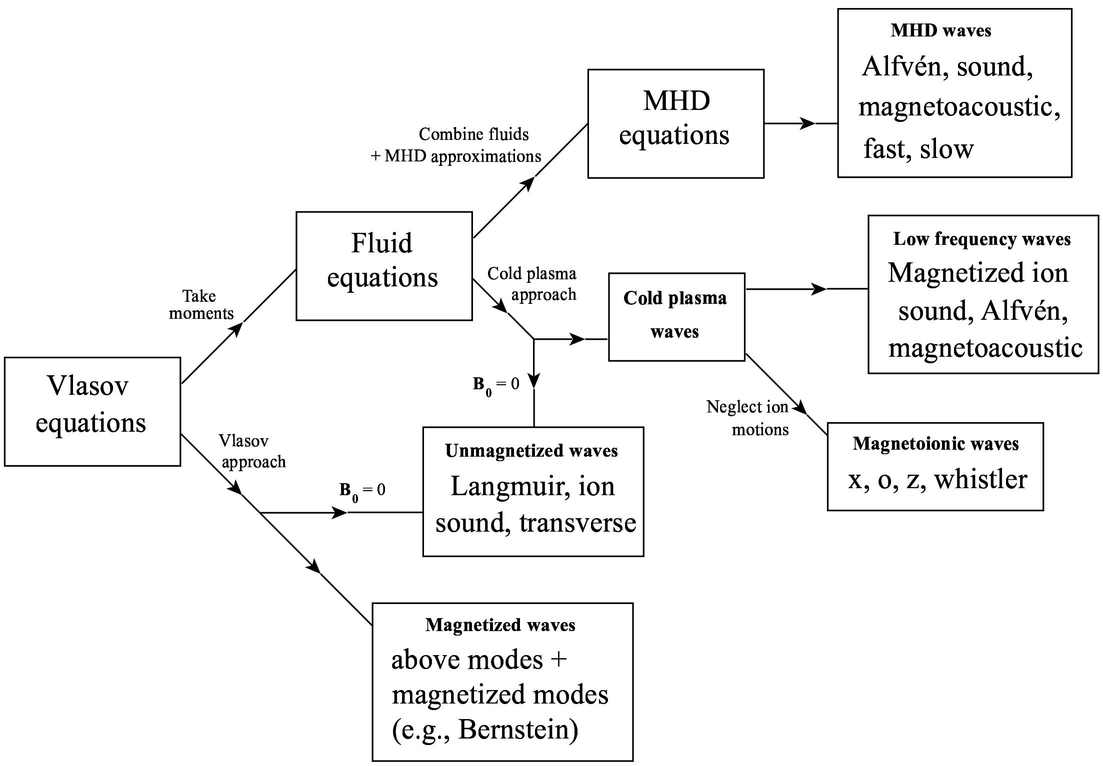

Figure 3.1: Flowchart illustrating the usual theoretical approaches to obtaining wave solutions from the Vlasov equations.

There are several theoretical approaches to studying waves in plasmas as illustrated in the flowchart in Figure 3.1. In Lecture 2 MHD wave solutions were obtained by linearizing the MHD equations. Averaging over particle velocities in MHD is a valid approximation so long as the relevant time scales are long in comparison with microscopic particle motion time scales ($\tau \gg \Omega_\alpha^{-1}$, ω_{pe}^{-1}, ν_e^{-1} and particle drift periods) and that spatial scale lengths are long in comparison with the Debye length and the thermal ion gyroradius. This is often not the case. In particular, the higher frequency electron motions typically do not obey these restrictions and so MHD is restricted to low frequencies less than the ion cyclotron and ion plasma frequencies.

An alternative approach is to start from the fluid equations (Section 2.3), but instead of combining the fluids and making the MHD approximations, the limit

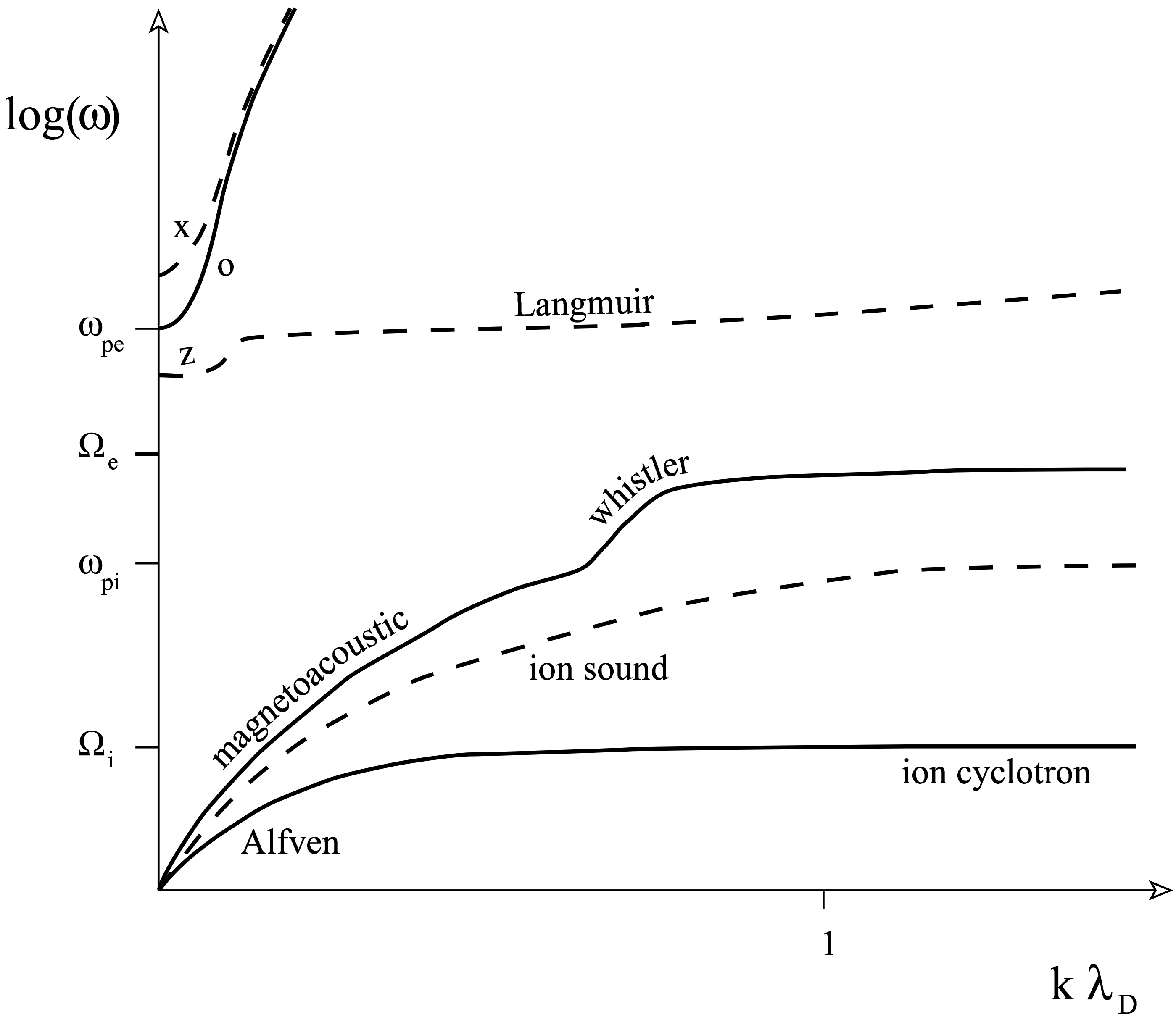

Figure 3.2: Dispersion relations in a weakly magnetized plasma for waves not propagating exactly parallel to **B**..

of zero plasma temperature is taken ($V_\alpha \to 0$). This yields the cold plasma wave solutions. If desired, finite pressures and temperatures can be included in fluid theory, as can several electron or ion components, so as to study the growth of associated waves by plasma instabilities.

The most general method, based on kinetic theory, starts directly from the Vlasov equations (3.11-3.17), without averaging over velocities. Using the Vlasov approach, generalized forms of all the wave modes derived via the other methods are found, along with new modes (for example, Bernstein modes near the cyclotron frequencies) which cannot be obtained from the more approximate methods.

Often it is necessary to use the Vlasov approach to treat wave damping and growth properly, the subject of Lecture 4. Kinetic effects are particularly important when waves interact resonantly with plasma particles, meaning that $\omega \approx \mathbf{k}.\mathbf{v}$ for the relevant plasma particles. However, damping can be added phenomenologically to the fluid equations (e.g., in the momentum equations) and non-resonant instabilities can be understood using fluid theories for multiple electron or ion components.

Figure 3.2 displays a schematic plot of the dispersion relations for the various wave modes in a weakly magnetized plasma ($\Omega_e \ll \omega_p$) that will be discussed in this lecture. In a weakly magnetized plasma the characteristic frequencies arranged in decreasing order are: the electron plasma frequency ω_{pe}, the electron cyclotron frequency Ω_e, the ion plasma frequency ω_{pi}, and the ion cyclotron frequency Ω_i. The MHD magnetoacoustic and Alfvén waves (see Lecture 2) are found in the bottom left-hand corner of Figure 3.2.

In this lecture we will concentrate on two particular paths in Figure 3.1. We first derive a general wave equation from Maxwell's equations. Then we take the Vlasov approach for unmagnetized plasmas and discuss the properties of Langmuir, ion sound and transverse waves. We then take the cold plasma approach (for magnetized plasmas) and ignore ion motions to derive the magnetoionic modes. Finally two examples of waves in space plasmas will be discussed.

3.4 General wave equation

In Section 2.6, MHD wave properties were obtained from the MHD equations assuming small amplitude plane wave solutions. This procedure is formalized using Fourier transforms. The Fourier transform $\tilde{F}(\omega, \mathbf{k})$ of any function $F(t, \mathbf{x})$ is defined by

$$\tilde{F}(\omega, \mathbf{k}) = \int \mathrm{d}t\mathrm{d}^3\mathbf{x}\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} F(t, \mathbf{x}) \,, \tag{3.1}$$

and its inverse transform by

$$F(t, \mathbf{x}) = \int \frac{\mathrm{d}\omega\mathrm{d}^3\mathbf{k}}{(2\pi)^4}\, e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} \tilde{F}(\omega, \mathbf{k}) \,. \tag{3.2}$$

Henceforth the tilde will be omitted and the Fourier transform identified by its arguments ω and $\mathbf{k}$. A useful property of the Fourier transform is that differential equations in real space (arguments t and $\mathbf{x}$) are converted into algebraic equations in Fourier space (arguments ω and $\mathbf{k}$), with

$$\frac{\partial F(t, \mathbf{x})}{\partial t} \to -i\omega F(\omega, \mathbf{k}) \,, \qquad \nabla F(t, \mathbf{x}) \to i\mathbf{k} F(\omega, \mathbf{k}) \,,$$
$$\boldsymbol{\nabla}\cdot\mathbf{G}(t, \mathbf{x}) \to i\mathbf{k}\cdot\mathbf{G}(\omega, \mathbf{k}) \,, \qquad \boldsymbol{\nabla}\times\mathbf{G}(t, \mathbf{x}) \to i\mathbf{k}\times\mathbf{G}(\omega, \mathbf{k}) \,. \tag{3.3}$$

Taking the Fourier transform of Maxwell's equations (2.11-2.14) yields the equations,

$$\mathbf{k} \times \mathbf{E}(\omega, \mathbf{k}) = \omega \mathbf{B}(\omega, \mathbf{k}) \,, \tag{3.4}$$

$$\mathbf{k}\times\mathbf{B}(\omega,\mathbf{k}) = -i\mu_0\mathbf{J}(\omega,\mathbf{k}) - \frac{\omega}{c^2}\mathbf{E}(\omega,\mathbf{k})\,, \tag{3.5}$$

$$\mathbf{k}\cdot\mathbf{E}(\omega,\mathbf{k}) = \frac{-i\rho(\omega,\mathbf{k})}{\varepsilon_0}\,, \tag{3.6}$$

$$\mathbf{k}\cdot\mathbf{B}(\omega,\mathbf{k}) = 0\,. \tag{3.7}$$

Equation (3.7) is redundant - the same result is obtained by taking the dot product of equation (3.4) with $\mathbf{k}$. From equation (3.4), one can write

$$\mathbf{B}(\omega,\mathbf{k}) = \frac{\mathbf{k}\times\mathbf{E}(\omega,\mathbf{k})}{\omega}\,, \tag{3.8}$$

and from equations (3.5) and (3.6), one can write

$$\rho(\omega,\mathbf{k}) = \frac{\mathbf{k}\cdot\mathbf{J}(\omega,\mathbf{k})}{\omega}\,. \tag{3.9}$$

Hence all the information concerning the fields is contained in the pair of quantities $\mathbf{E}(\omega,\mathbf{k})$ and $\mathbf{J}(\omega,\mathbf{k})$, and it is possible to regard $\mathbf{B}(\omega,\mathbf{k})$, $\rho(\omega,\mathbf{k})$ as subsidiary quantities. (Some authors prefer different combinations of the primary field and the field's source.) From equations (3.5) and (3.8) we obtain the wave equation,

$$\frac{c^2}{\omega^2}\mathbf{k}\times\{\mathbf{k}\times\mathbf{E}(\omega,\mathbf{k})\} + \mathbf{E}(\omega,\mathbf{k}) = \frac{-i\mu_0 c^2}{\omega}\mathbf{J}(\omega,\mathbf{k})\,. \tag{3.10}$$

and, in tensor form,

$$\left\{\frac{c^2}{\omega^2}k_ik_j + \left(1-\frac{k^2c^2}{\omega^2}\right)\delta_{ij}\right\}E_j(\omega,\mathbf{k}) = \frac{-i\mu_0 c^2}{\omega}J_i(\omega,\mathbf{k})\,. \tag{3.11}$$

Assuming that there are no externally applied currents, and that the current density $\mathbf{J}(\omega,\mathbf{k})$ is linear in $\mathbf{E}$, we can write

$$J_i(\omega,\mathbf{k}) = \sigma_{ij}(\omega,\mathbf{k})E_j(\omega,\mathbf{k})\,, \tag{3.12}$$

where σ_{ij} is the conductivity tensor. (Higher order contributions to J_i, corresponding to nonlinear processes, can be included - see section 4.7 below.) The conductivity tensor is replaced by the dimensionless dielectric tensor K_{ij}, with

$$K_{ij}(\omega,\mathbf{k}) = \delta_{ij} + \frac{i}{\varepsilon_0\omega}\sigma_{ij}(\omega,\mathbf{k})\,. \tag{3.13}$$

The dielectric tensor contains all the information concerning the linear response of the medium, and from it all relevant wave properties can be obtained (such as the dispersion relation, the polarization vector, and the ratio of electric to total energy). Equation (3.11) may be rewritten in the form

$$\Lambda_{ij}(\omega,\mathbf{k})E_j(\omega,\mathbf{k}) = 0\,, \tag{3.14}$$

for

$$\Lambda_{ij}(\omega,\mathbf{k}) = \frac{c^2}{\omega^2}(k_ik_j - k^2\delta_{ij}) + K_{ij}(\omega,\mathbf{k})\,. \tag{3.15}$$

Equation (3.14) consists of three simultaneous equations for the components of $\mathbf{E}$ (cf equation (2.46) for MHD waves). The condition for nontrivial solutions to exist is that the determinant of Λ_{ij} vanishes. This yields the *dispersion equation*

$$\Lambda(\omega,\mathbf{k}) = \det[\Lambda_{ij}(\omega,\mathbf{k})] = 0\,, \tag{3.16}$$

from which the dispersion relations for wave modes in a plasma are obtained. Note that a solution of the dispersion equation must satisfy each of the three simultaneous equations in (3.14). As in Lecture 2, this permits the polarization and other field characteristics of the mode to be determined (e.g., in what directions do the $\mathbf{E}$ and $\mathbf{B}$ fluctuations and variations in particle velocity, which represent the wave, vary). Subsequently these properties can be compared with data so as to identify the wave mode and thereafter use the wave to diagnose the properties of the plasma.

3.5 Vlasov approach for unmagnetized plasmas

In the Vlasov approach the plasma is assumed to be collisionless, and equations (2.9-2.16) are solved together. The particle velocity information is retained, in contrast to the fluid approach where particle velocities are averaged over. We proceed by linearizing and Fourier transforming the collisionless Boltzmann equation (2.9):

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{q}{m}\left(\mathbf{E}(t,\mathbf{x}) + \mathbf{v}\times\mathbf{B}(t,\mathbf{x})\right)\cdot\frac{\partial}{\partial \mathbf{v}}\right) f(\mathbf{v},t,\mathbf{x}) = 0\,, \tag{3.17}$$

where the distribution function is expanded in powers of the electric field:

$$f(\mathbf{v},t,\mathbf{x}) = f^{(0)}(\mathbf{v}) + \sum_{n=1}^{\infty} f^{(n)}(\mathbf{v},t,\mathbf{x})\,, \tag{3.18}$$

with $f^{(n)} \propto E^n$. Note that $\mathbf{B}$ is first order in $\mathbf{E}$ via (3.8 and that the unperturbed distribution function has no time or space dependence. After substituting (3.18) into (3.17) and retaining only linear terms in $\mathbf{E}$, we find

$$\frac{\partial f^{(1)}(\mathbf{v},t,\mathbf{x})}{\partial t} + \mathbf{v}\cdot\frac{\partial f^{(1)}(\mathbf{v},t,\mathbf{x})}{\partial \mathbf{x}} + \frac{q}{m}\left(\mathbf{E}(t,\mathbf{x}) + \mathbf{v}\times\mathbf{B}(t,\mathbf{x})\right)\cdot\frac{\partial f^{(0)}(\mathbf{v})}{\partial \mathbf{v}} = 0\,. \tag{3.19}$$

We now Fourier transform this equation to obtain

$$-i(\omega - \mathbf{k}\cdot\mathbf{v}) f^{(1)}(\mathbf{v},\omega,\mathbf{k}) + \frac{q}{m}\left(\mathbf{E}(\omega,\mathbf{k}) + \mathbf{v}\times\mathbf{B}(\omega,\mathbf{k})\right)\cdot\frac{\partial f^{(0)}(\mathbf{v})}{\partial \mathbf{v}} = 0\,. \tag{3.20}$$

The next step is to eliminate $\mathbf{B}$ using equation (3.8) and convert into tensor form, with

$$f^{(1)}(\mathbf{v},\omega,\mathbf{k}) = \frac{-iq}{m\omega(\omega - \mathbf{k}\cdot\mathbf{v})}\left\{(\omega - \mathbf{k}\cdot\mathbf{v})\delta_{sj} + k_s v_j\right\} E_j(\omega,\mathbf{k})\frac{\partial f^{(0)}(\mathbf{v})}{\partial v_s}\,. \tag{3.21}$$

The first order current density satisfies

$$\mathbf{J}^{(1)}(\omega,\mathbf{k}) = \sum_\alpha q \int \mathrm{d}^3\mathbf{v}\,\mathbf{v} f^{(1)}(\mathbf{v},\omega,\mathbf{k})\,, \tag{3.22}$$

and thus relates $\mathbf{J}^{(1)}$ to $\mathbf{E}$, using equation (3.21). Hence the dielectric tensor (3.13) takes the form

$$K_{ij} = \delta_{ij} + \sum_\alpha \frac{q^2}{m\varepsilon_0\omega^2}\int \mathrm{d}^3\mathbf{v}\frac{v_i}{(\omega - \mathbf{k}\cdot\mathbf{v})}\left\{(\omega - \mathbf{k}\cdot\mathbf{v})\delta_{sj} + k_s v_j\right\}\frac{\partial f^{(0)}(\mathbf{v})}{\partial v_s}\,. \tag{3.23}$$

The $(\omega - \mathbf{k}\cdot\mathbf{v})$ term in the denominator is a singularity, corresponding to resonance between particles and waves (where the component of the phase velocity of the wave parallel to $\mathbf{v}$ is equal to the particle speed v). As shown in the next Lecture this term leads to (kinetic) growth and damping of waves, as first proposed by Landau. Note, moreover, that the dielectric tensor depends on derivatives of the distribution function in velocity space. This leads to the growth and damping of waves due to anisotropies in velocity space.

For an unmagnetized plasma, K_{ij} may be separated into longitudinal and transverse components, with

$$K_{ij}(\omega,\mathbf{k}) = K_L(\omega,\mathbf{k})\kappa_i\kappa_j + K_T(\omega,\mathbf{k})(\delta_{ij} - \kappa_i\kappa_j)\,, \tag{3.24}$$

where $\boldsymbol{\kappa}$ is a unit vector along $\mathbf{k}$. Assuming that $f^{(0)}(\mathbf{v})$ is a Maxwellian distribution, we find

$$K_L(\omega, \mathbf{k}) = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 V^2} \int \mathrm{d}^3\mathbf{v} \frac{\omega}{(\omega - \mathbf{k}\cdot\mathbf{v})} \left(\frac{\mathbf{k}\cdot\mathbf{v}}{k}\right)^2 \frac{\exp(-v^2/2V^2)}{(2\pi)^{3/2}V^3}\,, \tag{3.25}$$

and

$$K_T(\omega, \mathbf{k}) = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 V^2} \int \mathrm{d}^3\mathbf{v} \frac{\omega}{(\omega - \mathbf{k}\cdot\mathbf{v})} \frac{1}{2} \left\{v^2 - \left(\frac{\mathbf{k}\cdot\mathbf{v}}{k}\right)^2\right\} \frac{\exp(-v^2/2V^2)}{(2\pi)^{3/2}V^3}\,, \tag{3.26}$$

with the plasma frequency $\omega_{p\alpha}^2 = n_\alpha q_\alpha^2/\varepsilon_0 m_\alpha$ for each species α. The $(\omega - \mathbf{k}\cdot\mathbf{v})$ term in the denominator is a singularity, corresponding to resonance between particles and waves (where the component of the phase velocity of the wave parallel to $\mathbf{v}$ is equal to the particle speed v). The integrals (3.25) and (3.26) are evaluated using complex analysis (choosing the contour of integration around each singularity in the sense that obeys causality, as first prescribed by Landau), and the final form of the dielectric tensor is

$$K_L(\omega, \mathbf{k}) = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2 V_\alpha^2} \{1 + y_\alpha Z(y_\alpha)\}\,, \tag{3.27}$$

$$K_T(\omega, \mathbf{k}) = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} y_\alpha Z(y_\alpha)\,, \tag{3.28}$$

where $y_\alpha = \omega/\sqrt{2}kV_\alpha$ and $Z(y)$ is called the *plasma dispersion function* (Fried and Conte), which is complex. For real ω, the plasma dispersion function satisfies

$$Z(y) = -2\exp(-y^2) \int_0^y \mathrm{d}t \exp(t^2) + i\sqrt{\pi}\exp(-y^2)\,. \tag{3.29}$$

For small y ($y^2 \ll 1$), the real part of $Z(y)$ has the following power series expansion:

$$\mathrm{Re}[Z(y)] \approx -2y + \frac{4y^3}{3} - \frac{8y^5}{15} + \ldots\,, \tag{3.30}$$

and for large y ($y^2 \gg 1$), $\mathrm{Re}[Z(y)]$ has the following asymptotic expansion:

$$\mathrm{Re}[Z(y)] \approx -\frac{1}{y} - \frac{1}{2y^3} - \frac{3}{4y^5} - \ldots\,. \tag{3.31}$$

The imaginary part of the dielectric tensor relates to (kinetic) wave damping and the resonant denominator ($\omega - \mathbf{k}.\mathbf{v}$ in the equations above, as discussed further in Lecture 4.

In general both real and imaginary parts of K_{ij} and the Fried-Conte function need to be retained and the dispersion equation (3.16) is a complex equation with

$$\Lambda_{ij} = \begin{bmatrix} [K_T(\omega,\mathbf{k})] - n^2 & 0 & 0 \\ 0 & [K_T(\omega,\mathbf{k})] - n^2 & 0 \\ 0 & 0 & [K_L(\omega,\mathbf{k})] \end{bmatrix}, \tag{3.32}$$

where n is the wave *refractive index*, with $n^2 = k^2c^2/\omega^2$. This equation is usually solved for complex ω as a function of real $\mathbf{k}$. Sometimes, however, in situations where evanescence or spatial growth/damping of waves appears most relevant physically, it is solved for real ω as a function of complex $\mathbf{k}$.

For the moment we are concerned only with finding dispersion relations, and so only need to retain the real part of K_{ij}. Then the dispersion equation (3.16) takes the form

$$\Lambda(\omega, \mathbf{k}) = (n^2 - \mathrm{Re}[K_T(\omega,\mathbf{k})])^2 \mathrm{Re}[K_L(\omega,\mathbf{k})] = 0\,. \tag{3.33}$$

There are three wave mode solutions to this equation, corresponding to transverse, Langmuir and ion sound waves.

Transverse electromagnetic waves

$(n^2 - \text{Re}[K_T(\omega, \mathbf{k})])^2 = 0$ is a double solution which corresponds to transverse waves in a plasma. The degeneracy is broken when magnetic effects are included (corresponding to magnetoionic x-mode and o-mode waves in Section 3.6). It can be shown that there are no solutions for transverse waves in the low phase speed limit ($y_\alpha^2 \ll 1$). In the high phase speed limit, with $y_\alpha^2 \gg 1$, after substituting $Z(y_\alpha) \approx -1/y_\alpha$ from equation (3.31) into $n^2 - \text{Re}[K_T(\omega, \mathbf{k})] = 0$, we obtain

$$\omega^2 = \omega_p^2 + k^2 c^2 , \tag{3.34}$$

which is the dispersion relation for transverse waves in a plasma. They are transverse because the electric field vector of the wave always lies in the plane perpendicular to $\mathbf{k}$. In the high-frequency limit $\omega \gg \omega_p$, these are equivalent to electromagnetic waves in a vacuum. There is a low frequency cutoff to the transverse wave dispersion relation at the plasma frequency. The characteristic response time for electrons in a plasma is ω_{pe}^{-1}, and so for wave frequencies $\omega < \omega_p$ electrons in the plasma can move fast enough to cancel out the electric field of the wave. Waves are reflected (or refracted through a large angle) at the cutoff frequency. An analogous effect occurs in metals, where the electrons in the conduction band behave like free electrons in a plasma. For metals the plasma frequency is at X-ray frequencies; for wave frequencies above the plasma frequency the metal is transparent, and wave frequencies below are reflected from the surface of the metal.

Langmuir waves

The dispersion relation for Langmuir waves is obtained by solving $\text{Re}[K_L(\omega, \mathbf{k})] = 0$ whilst ignoring the motion of the ions. Additionally we assume $y_e^2 \gg 1$, which corresponds to assuming $v_\phi \gg V_e$ (it turns out that Langmuir waves with $v_\phi \lesssim V_e$ are strongly Landau damped - see next lecture). The first three terms in the asymptotic expansion of the plasma dispersion function (3.31) are retained. The dispersion relation for Langmuir waves is

$$\omega^2 = \omega_p^2 + 3k^2 V_e^2 . \tag{3.35}$$

Langmuir waves are longitudinal electrostatic waves; i.e., the electric field vector of the wave is parallel to $\mathbf{k}$. (Question: why are these waves electrostatic?)

Ion sound waves

The dispersion relation for ion sound waves is also obtained by solving $\text{Re}[K_L(\omega, \mathbf{k})] = 0$. However, in this case, both the electron and ion contributions are included. Here we assume that there is only one ion species (protons) in the plasma. Further we assume that the wave phase velocity lies in the range $V_i \ll v_\phi \ll V_e$. Accordingly, the power series expansion (3.30) is adopted for the electron contribution to the plasma dispersion function and the asymptotic expansion (3.31) is used for the ion contribution. The resulting dispersion relation is

$$\omega = \frac{k v_s}{(1 + k^2 \lambda_{De}^2)^{1/2}} , \tag{3.36}$$

where $v_s = \sqrt{k_B(T_e + 3T_i)/m_i}$ is the ion sound speed, and $\lambda_{De} = V_e/\omega_{pe}$ is the Debye length. In the limit $k\lambda_{De} \ll 1$, which corresponds to $v_\phi \gg \omega_{pi} V_e/\omega_{pe}$ for $T_i \ll T_e$, this reduces to

$$\omega = k v_s , \tag{3.37}$$

which is the dispersion relation for ion sound waves. These are also longitudinal, electrostatic waves.

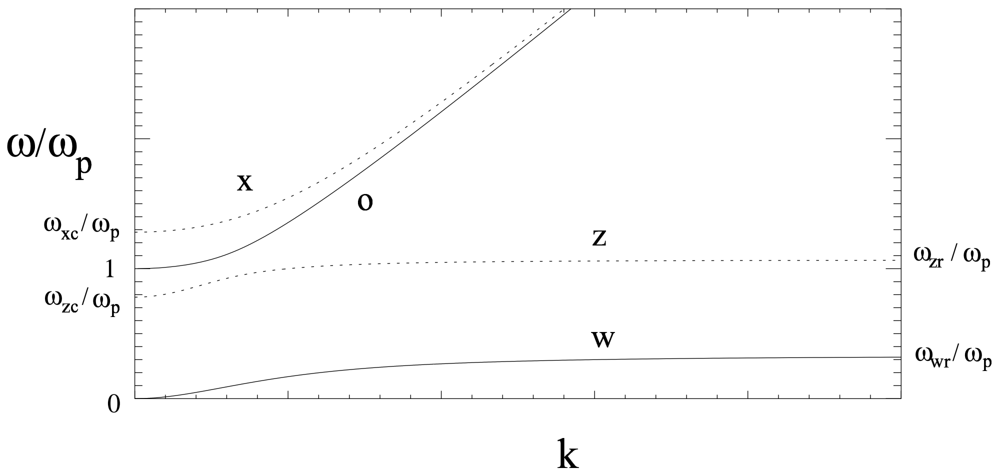

Figure 3.3: Dispersion relations for the magnetoionic modes for $\Omega_e < \omega_{pe}$.

3.6 The magnetoionic modes

The *cold plasma* approach to obtaining wave solutions is more general than the MHD approach outlined in Lecture 2, and less general than the Vlasov approach discussed in the previous section. The starting point is the fluid equations (Section 2.3). Other assumptions are that (i) different particle species do not influence each other through collisions, and (ii) the plasma is cold; i.e., $V_\alpha = 0$, so that each particle travels at the flow velocity for that particular species. The steps to derive the cold plasma dielectric tensor are: (i) Linearize the equation of fluid motion (linear in $\mathbf{E}$) and Fourier transform. (ii) Rewrite this equation as the flow velocity as a function of $\mathbf{E}$. (iii) Substitute this expression into equation (2.16) thus relating the current density $\mathbf{J}$ to the electric field $\mathbf{E}$. This yields the dielectric tensor. The interested reader is directed to the references for more specific details.

The solution to the cold plasma dispersion equation gives the cold plasma modes. These include generalized versions of ion sound waves and low-β MHD waves (Alfvén and magnetoacoustic waves). If ion motions are neglected, meaning that only solutions with $\omega \gg \Omega_{ci}, \omega_{pi}$ are sought, then the cold plasma modes further simplify to the *magnetoionic modes*. The relevant dielectric tensor is

$$K_{ij}(\omega) = \begin{bmatrix} (1-X-Y^2)/(1-Y^2) & iXY/(1-Y^2) & 0 \\ -iXY/(1-Y^2) & (1-X-Y^2)/(1-Y^2) & 0 \\ 0 & 0 & 1-X \end{bmatrix}, \quad (3.38)$$

for $X = \omega_{pe}^2/\omega^2$ and $Y = \Omega_e/\omega$. There are only two plasma parameters in the magnetoionic theory: ω_{pe} and Ω_e. Note that there is no k-dependence in the dielectric tensor; however, the waves remain dispersive since the dispersion equation depends on n and so the ratio kc/ω.

The dispersion equation (3.16) for magnetoionic waves then becomes a quadratic equation for n^2, thereby yielding two solutions for $n^2(\omega)$ that are commonly named the extraordinary and ordinary modes. Each solution comprises two branches, as shown in Figure 3.3. The lower branch of the extraordinary mode is the z-mode, with a low frequency cutoff at

$$\omega_{zc} = -\frac{1}{2}\Omega_e + \frac{1}{2}\left\{\Omega_e^2 + 4\omega_p^2\right\}^{1/2}, \quad (3.39)$$

and a resonance at

$$\omega_{zr}^2 = \frac{1}{2}(\omega_p^2 + \Omega_e^2) + \frac{1}{2}\left\{(\omega_p^2 + \Omega_e^2)^2 - 4\omega_p^2\Omega_e^2\cos^2\theta\right\}^{1/2} , \tag{3.40}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{B}$. The upper branch of the extraordinary mode is the x-mode, with a low frequency cutoff at

$$\omega_{xc} = \frac{1}{2}\Omega_e + \frac{1}{2}\left\{\Omega_e^2 + 4\omega_p^2\right\}^{1/2} . \tag{3.41}$$

The lower branch of the ordinary mode is the whistler mode, with a resonance at

$$\omega_{wr}^2 = \frac{1}{2}(\omega_p^2 + \Omega_e^2) - \frac{1}{2}\left\{(\omega_p^2 + \Omega_e^2)^2 - 4\omega_p^2\Omega_e^2\cos^2\theta\right\}^{1/2} . \tag{3.42}$$

The upper branch of the ordinary mode is the o-mode with a cutoff at $\omega_{oc} = \omega_p$. The o-mode and x-mode are the magnetized counterparts of unmagnetized transverse waves. They have opposite senses of polarization (the x-mode has the same handedness as a gyrating electron). In weakly magnetized plasmas, z-mode waves are the magnetized counterpart of Langmuir waves.

Atmospheric whistler waves

At low frequencies, whistler waves have the approximate dispersion relation $\omega \approx \Omega_e|\cos\theta|c^2k^2/\omega_p^2$. Hence the group speed satisfies

$$v_g = \frac{\partial\omega}{\partial k} = \frac{2\Omega_e|\cos\theta|c^2k}{\omega_p^2} , \tag{3.43}$$

and thus increases with increasing k (and with increasing ω - see Figure 3.3). Figure 3.4 shows an example of whistler waves excited over a broad range of frequencies by a lightning discharge, which are observed after they have propagated to the opposite hemisphere along terrestrial magnetic field lines. After the whistler waves have propagated some distance, the lower frequencies lag behind the higher frequencies due to their lower group speeds. This produces a falling tone over timescales of several seconds which can be detected by radio receivers. Figure 3.4 is an example of a *dynamic spectrum*. Electric and magnetic fluctuations associated with a wave are processed to produce Fourier spectra and plotted as power as a function of frequency and time.

Ionospheric sounding

A beam of radio waves directed from the ground towards the ionosphere is reflected back towards the ground. As the transverse electromagnetic waves enter the bottom of the ionosphere ($\omega_p = 0$) they separate into x-mode and o-mode components. As they propagate upwards through the ionosphere the local plasma frequency increases with increasing ionospheric plasma density. The wave frequency remains constant when time variations can be neglected (and the wavelength is significantly smaller than the scale height of the variation in plasma density in the ionosphere, so that an eigenfunction analysis can be avoided). This implies that k and the ratio ω/ω_p decrease with increasing height. Accordingly, as the waves propagate into the increasing plasma density, they effectively move downwards on the x-mode and o-mode dispersion curves in Figure 3.3. As the ratio $\omega/\omega_p(z)$ decreases with increasing altitude z, the waves first encounter the x-mode cutoff (where $\omega = \omega_{xc}(z)$), where the x-mode waves are reflected. The o-mode waves are reflected at a higher altitude (where $\omega = \omega_p(z)$). This is illustrated in Figure 3.5. At the ground receiver, there is

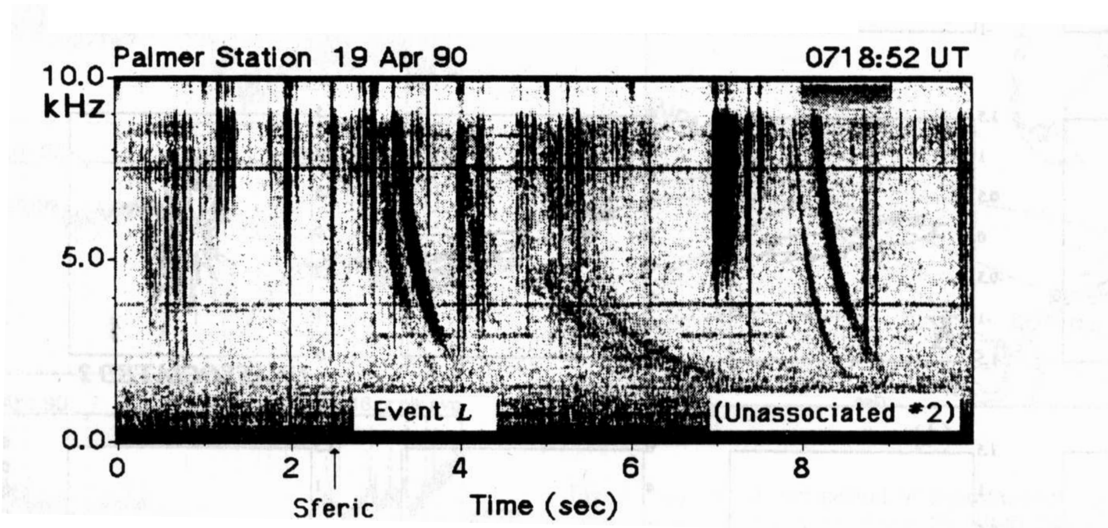

Figure 3.4: Examples of whistler waves (Burgess and Inan, 1993).

a time delay between the x-mode waves (which arrive first) and the o-mode waves. They are easily distinguished, due to their opposite senses of polarization. Thus, by measuring the echo and delay times as functions of frequency, radio waves can be used to probe the ionosphere and thus determine the plasma density profile with height.

3.7 Further reading:

1. Melrose, D. B., 1986, *Instabilities in space and laboratory plasmas*, Cambridge University Press, Cambridge.

2. Melrose, D. B. and McPhedran, R. C., 1991, *Electromagnetic processes in dispersive media*, Cambridge University Press, Cambridge.

3. Landau, L. D., 1946, On the vibrations of the electron plasma, *J. Phys. (USSR)*, **10**, 25.

4. Fried, B. D. and Conte, S. D., 1961, *The plasma dispersion function*, Academic Press, New York.

5. Burgess, W. C. and Inan, U. S., 1993, The role of ducted whistlers in the precipitation loss and equilibrium flux of radiation belt electrons, *J. Geophys. Res.*, **98**, 15643.

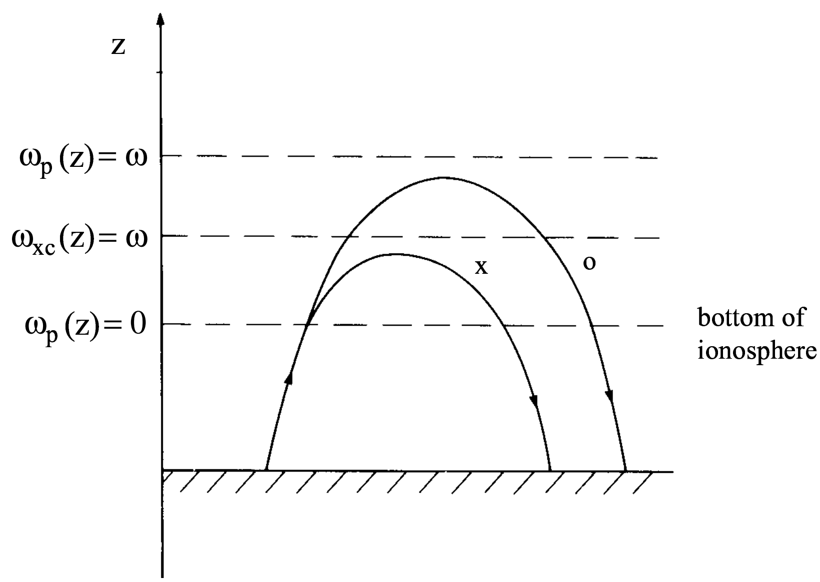

Figure 3.5: Reflection of radio waves in the ionosphere (from Melrose, 1986).