Lecture 2

Kinetic, Fluid & MHD Theories

The Vlasov equations are introduced as a starting point for both kinetic theory and fluid theory in a plasma. The equations of fluid theory are derived by taking moments of Boltzmann's equation. The one-fluid theory for a magnetized plasma is called *magnetohydrodynamics* (MHD). Various properties of the MHD equations are outlined with reference to the frozen-in flux condition, magnetic pressure and tension, and MHD waves.

2.1 Aims, Learning Outcomes, and General Considerations

Aims. To develop, justify, and understand the basic equations governing the bulk motion of plasmas in the kinetic, fluid, and MHD theories. These differ from orbit theory (Lecture 1) by being dynamical theories for plasma particles moving in both self-consistent and prescribed electromagnetic fields. They are required to understand the dynamics of general plasma flows (in solar system contexts and elsewhere), wave modes in plasmas, instabilities, particle acceleration, and shocks.

Expected Learning Outcomes. You are expected to

- Know the benefits and disadvantages of kinetic, fluid, and MHD descriptions of plasmas, both from one another and from orbit theory.
- Be able to write down and explain the terms in the governing equations of kinetic theory.
- Be able to list the governing equations, and explain the terms therein, for fluid theories.
- Be able to derive the fluid equations corresponding to conservation of mass, charge, and momentum, and to explain the terms in the associated energy equation.
- Be able to identify the MHD equations and to derive the associated mass and momentum conservation equations.
- Be able to identify the terms in the MHD version of Ohm's Law and to use the equation to explain convection electric fields and frozen-in magetic fields.
- Understand magnetic pressure and tension forces.

- Outline the derivation of the dispersion equation for the 3 basic MHD wave modes and describe their properties.

Preamble Orbit theory (lecture 1) describes the motion of individual test particles in prescribed $\mathbf{E}$ and $\mathbf{B}$ fields. However, it is not self-consistent, since the feedback to the EM fields is not included for the currents and charge separations induced by particle drifts. It is also not a dynamic theory. Kinetic theories treat all the plasma particles simultaneously by evolving the (single) particle distribution function in the prescribed and (ideally) self-consistent electromagnetic fields. Fluid theories integrate or average over the distribution function to treat the bulk plasma motion and properties.

2.2 Distribution functions

The particle distribution function $f(\mathbf{v}, \mathbf{x}, t)$ is defined so that the total number of particles in a differential six-dimensional phase space element $\mathrm{d}^3\mathbf{v}\mathrm{d}^3\mathbf{x}$ is equal to $f(\mathbf{v}, \mathbf{x}, t)\mathrm{d}^3\mathbf{v}\mathrm{d}^3\mathbf{x}$. The particle number density (number of particles per unit volume) is

$$n(\mathbf{x}, t) = \int \mathrm{d}^3\mathbf{v}\, f(\mathbf{v}, \mathbf{x}, t) \tag{2.1}$$

Other physical quantities are obtained by taking moments, where the moment of quantity $\Theta(\mathbf{v})$ is defined by

$$\langle\Theta(\mathbf{v})\rangle = \frac{1}{n(\mathbf{x}, t)} \int \mathrm{d}^3\mathbf{v}\, \Theta(\mathbf{v}) f(\mathbf{v}, \mathbf{x}, t) \tag{2.2}$$

Using (2.2), the following physical quantities are defined for particle species α:

- Fluid velocity:
$$\mathbf{u}_\alpha = \langle\mathbf{v}_\alpha\rangle \tag{2.3}$$

- Mean thermal velocity:
$$V_\alpha = \sqrt{\langle(\mathbf{v}_\alpha - \mathbf{u}_\alpha)^2\rangle} \tag{2.4}$$

- Mass density:
$$\eta = \sum_\alpha m_\alpha n_\alpha \tag{2.5}$$

- Mean mass velocity:
$$\mathbf{U} = \frac{1}{\eta}\sum_\alpha m_\alpha n_\alpha \mathbf{u}_\alpha \tag{2.6}$$

- Pressure tensor:
$$p_{\alpha,ij} = m_\alpha n_\alpha \langle w_{\alpha,i} w_{\alpha,j}\rangle \tag{2.7}$$

  where $\mathbf{w}_\alpha = \mathbf{v}_\alpha - \mathbf{U}$ is the velocity of a particle relative to the mean mass velocity.

The velocity distribution function for a plasma in thermal equilibrium is a *Maxwellian*, with

$$f(\mathbf{v}) = \frac{n}{(2\pi)^{3/2} m^3 V^3} \exp\left[-\frac{v^2}{2V^2}\right] . \tag{2.8}$$

Other distributions are often detected in space plasmas; for example, bi-Maxwellians with different temperatures in directions parallel and perpendicular to the background magnetic field, and generalized Lorentzian (or Kappa) distributions which depart from the Maxwellian functional form at high energies and instead obey a power law.

2.3 Basic equations

Boltzmann's equation

The distribution function $f_\alpha(\mathbf{v}, \mathbf{x}, t)$ for species α satisfies

$$\frac{df_\alpha}{dt} = \frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{x}} + \frac{q_\alpha}{m_\alpha}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B} + \frac{m_\alpha}{q_\alpha}\mathbf{g}\right) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\text{coll.}} + \left(\frac{\partial f_\alpha}{\partial t}\right)_{\text{oth}} . \tag{2.9}$$

The left-hand side is equal to the total time derivative (in six-dimensional phase space) of the distribution function. The terms on the right-hand side treat collisional effects and any other effects such as charge-exchange collisions, ionization, chemical reactions, scattering by waves etc.

Usually the spatial resolution for such calculation is restricted to scales greater than the *Debye length* λ_D. This is the scale over which the long-range electrostatic potential of each particle is shielded by other (oppositely charged) particles in the plasma. Consider a test particle (proton) placed in a plasma. Electrons are attracted to the test particle (and ions repelled, but they are less mobile due to their relatively large mass), and form a "sheath" around the proton, so that the potential for the test particle has the form,

$$\phi \propto \frac{1}{r} \exp\left(-\frac{r}{\lambda_D}\right) , \tag{2.10}$$

where the exponential factor is due to Debye shielding. The Debye length $\lambda_{D\alpha} = V_\alpha/\omega_p$, where ω_p is the electron plasma frequency, with $\omega_p^2 = n_e e^2/m_e \varepsilon_0$.

Liouville's Equation is a special version of the Boltzmann equation in which the righthand side equals zero, and so the number of particles in a given phase space volume remains constant (although the coordinates of the phase space volume may change as the particles move under the influence of forces). Liouville's equation is very useful in treating the evolution of test particles. However, it is not fully self-consistent since it does not include, for instance, the effects of growing waves scattering particles. The Boltzmann equation is arguably the most general kinetic equation available. However, it is very demanding computationally for realistic applications, thereby often requiring different approaches to be used.

Maxwell's equations

$$\nabla \times \boldsymbol{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's Law)} \tag{2.11}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \text{(Ampere's Law)} \tag{2.12}$$

$$\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0 \qquad \text{(Poisson's equation)} \tag{2.13}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{2.14}$$

Charge and current densities

$$\rho(\mathbf{x},t) = \sum_\alpha q_\alpha n_\alpha = \sum_\alpha \int d^3\mathbf{v} q_\alpha f_\alpha(\mathbf{v},\mathbf{x},t) \tag{2.15}$$

$$\mathbf{J}(\mathbf{x},t) = \sum_\alpha q_\alpha n_\alpha \langle \mathbf{v}_\alpha \rangle = \sum_\alpha \int d^3\mathbf{v} q_\alpha \mathbf{v} f_\alpha(\mathbf{v},\mathbf{x},t) \tag{2.16}$$

Vlasov equations and Kinetic Theory

If Boltzmann's equation (2.9) is solved in situations where **E** and **B** are known external fields then it is a linear differential equation. However in a plasma, governed by the set of equations (2.9) - (2.16), one must solve for *self-consistent* **E** and **B** fields. The equations that describe how charge and current densities affect the magnetic and electric fields (Maxwell's equations) must also be considered. The interdependent nature of the particle and field interactions is illustrated schematically in Figure 2.1. The velocity of a charged particle injected into a plasma will change under the influence of the existing **E** and **B** fields. These forces are different for electrons and ions, whose subsequent motion alters the charge distribution and induces currents, which in turn alter the fields. When equations (2.9) - (2.16) are solved in a self-consistent manner, with the collisional term in Boltzmann's equation set to zero, they are referred to as the Vlasov equations.

Equations (2.9) - (2.16) are a system of nonlinear integro-differential equations. They provide the basis for most plasma kinetic theories (specifically including those treated in later lectures) and fluid theory.

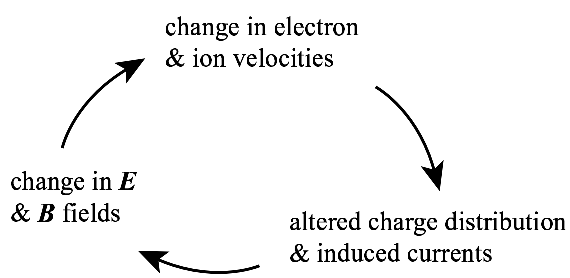

Figure 2.1: Flow chart illustrating the nonlinear interactions between particles and electromagnetic fields in a plasma.

2.4 Fluid theory

In the fluid description, information on the particle velocity distribution is replaced by values "averaged" over velocity space. This approximation is justified provided that the relevant time scales are long in comparison with microscopic particle motion time scales ($\tau > \Omega_e^{-1}$, ω_p^{-1}, ν_e^{-1}, where ν_e is the collisional frequency) and that spatial scale lengths are long in comparison with the Debye length and the thermal ion gyroradius. There is no requirement that the particle distribution be Maxwellian. Instead, the required "averaging" over velocity space is performed by taking moments of Boltzmann's equation (2.9). Moment equations are obtained by multiplying (2.9) by an arbitrary function of velocity $\Theta(\mathbf{v})$ and integrating over velocity space. Note that if the particle velocity distribution is in truth composed of several components with widely different parameters then several "fluid" components can be introduced to approximate the evolution of the total distribution function.

Conservation of particles: Taking the zeroth order moment, with $\Theta(\mathbf{v}) = 1$, gives

$$\frac{\partial n_\alpha}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(n_\alpha \mathbf{u}_\alpha) = 0\,. \tag{2.17}$$

The right-hand side is zero due to particle conservation for ions and electrons (for an ideal plasma, ignoring ionization, recombination and charge exchange effects); i.e.,

$$\left(\frac{\partial n_p}{\partial t}\right)_{\text{coll.}} = 0\ , \qquad \left(\frac{\partial n_e}{\partial t}\right)_{\text{coll.}} = 0\ . \tag{2.18}$$

Mass conservation equation: multiply (2.17) by m_α and sum over α:

$$\frac{\partial \eta}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\eta \mathbf{U}) = 0\,. \tag{2.19}$$

Charge conservation equation: Multiply (2.17) by q_α and sum over α:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\eta \mathbf{J}) = 0\,. \tag{2.20}$$

This equation can be derived directly from Poisson's Equation and Ampere's Law.

Momentum conservation equation or force balance: We take the first order moment $[\Theta(m\mathbf{v}) = m\mathbf{v}]$ by multiplying (2.9), rewritten in tensor form, by mv_s and integrating over velocity space. After some manipulation, this is expressed in the form (omitting particle species α),

$$\begin{aligned}\frac{\partial}{\partial t}(mnu_s) + \frac{\partial}{\partial x_i}\left[p_{is} + mn(U_i u_s + U_s u_i - U_i U_s)\right]&\\ -nqE_s - nq\epsilon_{sjk}u_j B_k - nm_\alpha g_s = \pm P_s&\end{aligned} \tag{2.21}$$

Here $\mathbf{P}$ is the momentum density, with

$$\mathbf{P} = m_p \int \mathrm{d}^3\mathbf{v}_p\, \mathbf{v}_p \left(\frac{\partial f_p}{\partial t}\right)_{\text{coll.}} = -m_e \int \mathrm{d}^3\mathbf{v}_e\, \mathbf{v}_e \left(\frac{\partial f_e}{\partial t}\right)_{\text{coll.}}\ , \tag{2.22}$$

since collisions between electrons and ions within the plasma do not change the total momentum density of the system.

Energy conservation equation: The second order moment of Boltzmann's equation yields the equation of energy continuity. It is not quoted here. However, the MHD version of this equation is given below as (2.31).

2.5 MHD equations

Magnetohydrodynamic theory involves a further simplification of fluid theory, where the proton and electron fluids are combined and assumed to possess a common flow velocity U. The *equation of motion* for the MHD fluid is derived by adding electron and proton forms of (2.21), to give

$$\frac{\partial}{\partial t}(\eta U_s) + \frac{\partial}{\partial x_i}(p_{is} + \eta U_i U_s) - \rho E_s - \varepsilon_{sjk} j_j B_k - \eta g_s = 0\,, \tag{2.23}$$

where $p_{is} = p_{p,is} + p_{e,is}$. We assume that the distribution of particle velocities is sufficiently random such that the pressure tensor may be approximated by a scalar, with $p_{ij} = p\delta_{ij}$. In vector form, the MHD equation of motion is,

$$\eta\left[\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\boldsymbol{\cdot}\boldsymbol{\nabla})\mathbf{U}\right] = -\nabla p + \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} + \eta\mathbf{g}\,. \tag{2.24}$$

The equation for conservation of mass density was used to obtain this form. Usual MHD theories assume $\rho = 0$ and ignore gravity to arrive at the form

$$\eta \left[\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \boldsymbol{\nabla}) \mathbf{U} \right] = -\nabla p + \mathbf{J} \times \mathbf{B} \,. \tag{2.25}$$

The fluid velocity $\mathbf{U}(\mathbf{x}, \mathbf{t})$ is an *Eulerian* velocity, which refers to the velocity of a fluid element, and not the the velocity of individual particles that constitute that fluid element at any one time. This is to be contrasted with a *Lagrangian* velocity, which is the time derivative of the position vector of a particle, and is thus only a function of time; e.g., Newton's equation of motion for a single particle is Lagrangian. The quantity in square brackets on the left-hand side of (2.25) is called the *convective derivative*.

A further relation, linking $\mathbf{J}$ and the fields, is obtained by multiplying the proton form of (2.21) by $-e/m_p$ and the electron form of (2.21) by e/m_e (where e is the charge of an electron). Terms quadratic in velocity are ignored ensuring that the resulting expression will be linear in $\mathbf{J}$. After adding the two equations and making the following simplifying approximations (given that $m_e \ll m_p$):

$$n_e \approx n_p \approx \frac{\eta}{m_p} \,, \quad u_{p,s} \approx U_s \,, \quad u_{e,s} \approx U_s - \frac{m_p c}{\eta e} J_s \,,$$

and assuming that the momentum exchanged between electrons and ions is proportional to the relative velocity of the two types of particles, with

$$P_s = -\frac{\eta e J_s}{m_p \sigma} \,, \tag{2.26}$$

where σ is the conductivity coefficient, with $\sigma = \varepsilon_0 \omega_p^2 / \nu_e$, the generalized form of *Ohm's law* is

$$\mathbf{J} + \frac{\sigma m_p m_e}{\eta e^2} \frac{\partial \mathbf{J}}{\partial t} + \frac{\sigma m_p}{\eta e} \mathbf{J} \times \mathbf{B} = \sigma \left(\mathbf{E} + \mathbf{U} \times \mathbf{B} + \frac{m_p}{\eta e} \nabla p \right) \,. \tag{2.27}$$

The second term on the lefthand side is usually called the "inertia" term, while the third is called the "Hall" term. For low-frequency disturbances, with characteristic frequency $\omega \ll \nu_e$, the inertia term is proportional to ω/ν_e and may be dropped. In situations where the electron cyclotron frequency $\Omega_e \ll \nu_e$, the Hall term is proportional to Ω_{ce}/ν_e and may also be dropped. Note that this term remains very important when the ion and electron flows differ, for example near the center of magnetic reconnection regions and associated current sheets. If the pressure gradient term is also insignificant, then (2.27) reduces to

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{U} \times \mathbf{B}) \,. \tag{2.28}$$

In the perfectly conducting limit ($\sigma = \infty$), (2.28) further simplifies to

$$\mathbf{E} + \mathbf{U} \times \mathbf{B} = 0 \,. \tag{2.29}$$

This means that in a highly conducting plasma with a flow and zero current must set up an electric field $\mathbf{E} = -\mathbf{U} \times \mathbf{B}$. This so-called *convection electric field* leads to $\mathbf{E} \times \mathbf{B}$ drift of the plasma perpendicular to $\mathbf{B}$ (see Lecture 1 for examples). Equation (2.29) also leads to the situation of frozen-in magnetic flux, in which a plasma carries a magnetic field along with it. This condition may be stated formally as *the magnetic flux through a closed loop that moves with the fluid is constant in time*, where the magnetic flux $\Phi = \int \mathbf{B} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$, where $\hat{\mathbf{n}}$ is the unit normal to a surface $\mathbf{S}$. This is illustrated in Figure 2.2 for a closed loop at two consecutive

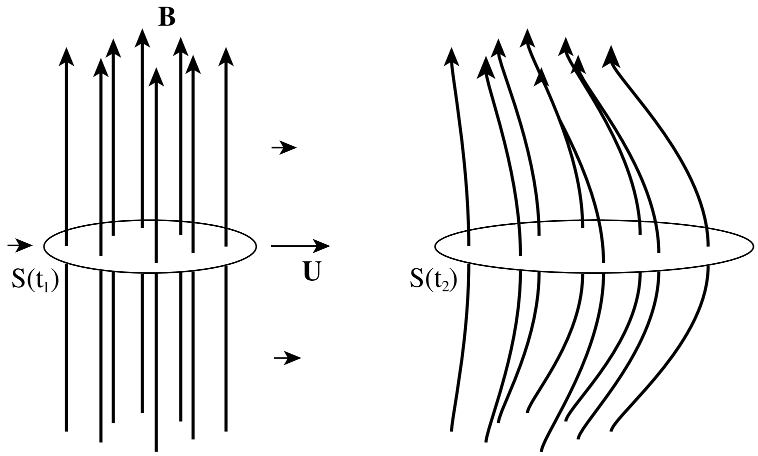

Figure 2.2: The closed loop S embedded in the fluid is stretched out at a later time $t_2 > t_1$ by a non-uniform fluid velocity profile. The magnetic flux through S remains constant and the field lines are tied to the fluid.

times t_1 and t_2, where the loop is stretched out as the fluid locally expands and the density of magnetic field lines enclosed by the loop decreases so as to conserve magnetic flux. The frozen-in flux condition $d\Phi/dt = 0$ can be proven by substituting (2.29) into Faraday's Law (2.11) to give

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{U} \times \mathbf{B}) \, , \tag{2.30}$$

and using Gauss's law and Stokes' theorem.

A *magnetic flux tube* is the surface generated by moving any closed loop parallel to the magnetic field lines it intersects at any given time. This surface encloses a constant amount of magnetic flux. As a consequence of flux conservation, the same fluid elements constitute a flux tube at different times; i.e., the fluid and magnetic field lines move together. A further consequence of the frozen-in flux condition is that all particles initially in a flux tube will remain in the same flux tube at later times.

An equation of energy continuity is derived by taking the second order moment of Boltzmann's equation, to give

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\eta U^2 + \frac{p}{\Gamma - 1} + \frac{B^2}{2\mu_0} + \frac{1}{2}\varepsilon_0 E^2\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{2}\eta U^2 \mathbf{U} + \frac{\Gamma}{\Gamma - 1} p\mathbf{U} + \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}\right) = 0 \, . \tag{2.31}$$

where Γ is the adiabatic index which takes the value 5/3 for a monatomic gas. Equation (2.31) assumes the adiabatic equation of state, for which there is no change in internal energy of a fluid element as it propagates, with

$$p \propto \eta^{\Gamma} \, . \tag{2.32}$$

MHD approximations

The following approximations are often made to produce a tractable set of equations:

1. The displacement current term in Ampere's law (2.12) is omitted. This approximation is sometimes called the "Darwin approximation" – an amusing aside is that it corresponds to slow temporal evolution. This can be justified

by comparing the LHS of (2.12) with the displacement current term:

$$|\boldsymbol{\nabla} \times \boldsymbol{B}| \approx \frac{B}{L}\,, \quad \frac{1}{c^2}\left|\frac{\partial \mathbf{E}}{\partial t}\right| \approx \frac{E}{c^2\tau}\,,$$

where L and τ are the characteristic MHD length and time scales. Thus,

$$\frac{|\partial \mathbf{E}/\partial t|\,/c^2}{|\boldsymbol{\nabla} \times \boldsymbol{B}|} \approx \left(\frac{L}{\tau}\right)^2 \frac{1}{c^2} \ll 1\,,$$

where $E/B \approx L/\tau$ from Faraday's law (2.11). Hence (2.12) reduces to

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \mathbf{J}\,. \tag{2.33}$$

2. Charge neutrality ($\rho = 0$) is typically satisfied in a plasma because the forces associated with any unbalanced charges imply a potential energy per particle that well exceeds the mean thermal energy per particle. The charge conservation equation (2.20) then reduces to

$$\boldsymbol{\nabla}\cdot\mathbf{J} = 0\,. \tag{2.34}$$

This also follows by taking the divergence of equation (2.33).

3. The approximation (2.28) or (2.29) to Ohm's law is assumed.

Final MHD equations

The *induction equation* is derived by eliminating $\mathbf{E}$ from Faraday's Law (2.11) and Ohm's Law (2.28), using (2.14), (2.33) and a vector identity:

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{U} \times \mathbf{B}) + \frac{1}{\mu_0\sigma}\nabla^2\mathbf{B}\,. \tag{2.35}$$

$\mathbf{E}$ has now been eliminated and $\rho = 0$, so Poisson's equation (2.13) does not contribute to the final set of equations. Equation (2.14) is effectively a boundary condition, since if $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ initially, then taking the divergence of (2.11) implies that $\boldsymbol{\nabla}\cdot\mathbf{B}$ remains zero henceforth.

It is remarked that (2.35) is often called a "dynamo" equation, since it can describe the generation of magnetic fields by magnetic dynamos. Specifically, the time evolution of $\mathbf{B}$ is related to spatial gradients in a velocity field $\mathbf{U}$ and $\mathbf{B}$ (often related to turbulent fields) acting against a diffusive term.

After making the above approximations, the final set of MHD equations are the induction equation and equations (2.19), (2.25) with $\rho = 0$, and (2.33):

$$\frac{\partial \eta}{\partial t} + \boldsymbol{\nabla}\cdot(\eta\mathbf{U}) = 0\,, \tag{2.36}$$

$$\eta\left[\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\cdot\boldsymbol{\nabla})\mathbf{U}\right] = -\nabla p + \mathbf{J} \times \mathbf{B} + \eta\mathbf{g}\,. \tag{2.37}$$

We now have one scalar and two vector equations in two scalar quantities (η, p) and two vector quantities ($\mathbf{B}$, $\mathbf{U}$). We thus require one more scalar equation to close the set of equations. This can either be the energy conservation equation (2.31) or, as is commonly adopted, an equation of state for the fluid; in this case the adiabatic equation of state (2.32).

2.6 Magnetic pressure and tension

The magnetic force (per unit volume) in the equation for fluid motion (2.25) may be re-expressed as

$$\mathbf{J}\times\mathbf{B} = \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} = -\boldsymbol{\nabla}\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\mathbf{B}\,. \tag{2.38}$$

The first term corresponds to the *magnetic pressure*, with $p_B = B^2/(2\mu_0)$. An important diagnostic of a plasma is the *plasma beta*, defined as the ratio of plasma thermal pressure to the magnetic pressure:

$$\beta = \frac{p}{B^2/2\mu_0}\,. \tag{2.39}$$

The second term can be further decomposed into two terms:

$$\frac{1}{\mu_0}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\mathbf{B} = \frac{B}{\mu_0}(\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla})(B\hat{\mathbf{b}}) = \hat{\mathbf{b}}\;\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\left(\frac{B^2}{2\mu_0}\right) + \frac{B^2}{\mu_0}\frac{\hat{\mathbf{n}}}{R_c}\,, \tag{2.40}$$

where $\hat{\mathbf{b}}$ is a unit vector in the direction of $\mathbf{B}$ and $\hat{\mathbf{n}}$ is the normal pointing towards the centre of curvature, defined by $(\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla})\hat{\boldsymbol{b}} = \hat{\mathbf{n}}/R_c$, where R_c is the radius of curvature of the field line. The first term cancels out the magnetic pressure gradient term in (2.38) in the direction along the field lines. This implies that the magnetic pressure force is not isotropic; only perpendicular components of ∇p_B exert force on the plasma. The second term in (2.40) corresponds to the *magnetic tension* force which is directed towards the centre of curvature of the field lines and thus acts to straighten out the field lines. A suitable analogy is the tension force transferred to an arrow by the stretched string of a bow. In this case the tension force pushes the plasma in the direction that will reduce the length of the field lines. Put another way, (2.38) becomes

$$\mathbf{J}\times\mathbf{B} = -\boldsymbol{\nabla}_\perp\left(\frac{B^2}{2\mu_0}\right) + \frac{B^2}{\mu_0}\frac{\hat{\mathbf{n}}}{R_c}\;. \tag{2.41}$$

2.7 MHD waves

For low-β plasmas, with $\beta \ll 1$ (also referred to as cold plasmas) the stresses in the plasma are predominantly magnetic. We seek MHD wave solutions in a cold magnetized plasma. In treating small-amplitude waves, the MHD equations are linearized, keeping only terms linear in the amplitude of the wave ($\mathbf{B}_1$, η_1, and $\mathbf{U}_1$). We seek plane wave solutions; i.e., solutions that vary in space and time as $\exp[-i(\omega t - kx)]$ (assuming that the plane wave propagates in the x-direction, with $\mathbf{k} = k\hat{\mathbf{x}}$). Additional assumptions are that the background magnetic field $\mathbf{B}_0$ and plasma density η_0 are uniform, that there are no background currents or electric fields, and that there is no bulk fluid motion. Our starting equations are:

$$\frac{\partial\eta}{\partial t} + \boldsymbol{\nabla}\cdot(\eta\mathbf{U}) = 0\,, \tag{2.42}$$

$$\eta\frac{\partial\mathbf{U}}{\partial t} = \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}\,, \tag{2.43}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{U}\times\mathbf{B})\,. \tag{2.44}$$

After linearizing, and replacing the time and spatial derivatives by $\partial/\partial t \rightarrow -i\omega$ and $\partial/\partial x \rightarrow ik$ (which corresponds to considering the Fourier transformed quantities in the plasma), these equations become

$$\omega\eta_1 - k\eta_0 U_{1x} = 0 \tag{2.45}$$

$$\omega\eta_0 \mathbf{U}_1 - k(\hat{\mathbf{x}}(\mathbf{B_1}\cdot\mathbf{B_0}) - B_{0x}\mathbf{B_1})/\mu_0 = 0\,, \tag{2.46}$$

$$\omega\mathbf{B}_1 + k(B_{0x}\mathbf{U}_1 - U_{1x}\mathbf{B}_0) = 0\,. \tag{2.47}$$

Without loss of generality we assume that $\mathbf{B}_0$ lies in the $x-z$ plane, with $\mathbf{B}_0 = (B_0\cos\theta, 0, B_0\sin\theta)$, where θ is the angle between $\mathbf{B}_0$ and $\mathbf{k}$. After eliminating $\mathbf{B}_1$ from (2.46) and (2.47), the three equations relating components of $\mathbf{U}_1$ are written in the following matrix form:

$$\begin{bmatrix} \left(\omega^2/k^2 - v_A^2\sin^2\theta\right) & 0 & v_A^2\sin\theta\cos\theta \\ 0 & \left(\omega^2/k^2 - v_A^2\cos^2\theta\right) & 0 \\ v_A^2\sin\theta\cos\theta & 0 & \left(\omega^2/k^2 - v_A^2\cos^2\theta\right) \end{bmatrix} \begin{bmatrix} U_{1x} \\ U_{1y} \\ U_{1z} \end{bmatrix} = \mathbf{0}\,, \tag{2.48}$$

where the Alfvén velocity v_A satisfies

$$v_A = \left(\frac{B_0^2}{\mu_0\eta_0}\right)^{1/2}\,. \tag{2.49}$$

The characteristics of the wave modes are obtained as solutions to this dispersion matrix, and to the so-called dispersion equation resulting from expanding the determinant of the matrix. Specifically, each wave mode has a dispersion relation $\omega(\mathbf{k})$ that satisfies the dispersion equation. Moreover, the solution must simultaneously satisfy each of the subsidiary equations. This means that the product of each row of the matrix with $\mathbf{U}$ must equal zero. This can be used to constrain how $\mathbf{U}$ varies for each wave mode, thereby describing the characteristic fluid motions that make up the wave.

A solution for $\mathbf{U}_1$ exists only if the determinant of this matrix vanishes. This yields two independent non-trivial solutions for ω as a function of k (known as the *dispersion relation*):

$$\omega^2 = k^2 v_A^2\cos^2\theta\,, \quad \omega^2 = k^2 v_A^2\,. \tag{2.50}$$

The first solution corresponds to *Alfvén waves*. After substituting the dispersion relation back into the matrix equation (2.48), we find that a solution for $\mathbf{U}_1$ is only possible if $U_{1x} = U_{1z} = 0$. Thus Alfvén waves are *shear* waves that shift plasma in the direction perpendicular to the plane containing the wavevector $\mathbf{k}$ and the background magnetic field $\mathbf{B}_0$, and that propagate with a phase velocity $v_\phi = \omega/k = v_A\cos\theta$. The wave motion in an Alfvén wave may be attributed to an interplay between magnetic tension and plasma inertia. When a fluid element is displaced relative to $\mathbf{B}_0$ the magnetic field is displaced with the fluid. The field line becomes locally curved, which generates a tension force tending to straighten out the field line. The inertia of the plasma causes it to overshoot, setting up an oscillatory motion. The density of the fluid is unaffected by the propagating Alfvén wave [$U_{1x} = 0 \Rightarrow \eta_1 = 0$ in (2.45)], and thus Alfvén waves are *incompressible.* The group velocity (velocity at which information propagates and the direction for energy flow) for Alfvén waves satisfies

$$\mathbf{v}_g = \left(\frac{\partial\omega}{\partial k_x}, \frac{\partial\omega}{\partial k_y}, \frac{\partial\omega}{\partial k_z}\right) = v_A\hat{\mathbf{b}}\,, \tag{2.51}$$

so that the flow of energy associated with Alfvén waves is directed along the background magnetic field direction.

The dispersion relation $\omega^2 = k^2 v_A^2$ corresponds to the *magnetoacoustic mode*. Substituting the dispersion relation into (2.48) yields the requirement that $U_{1y} = 0$ (for $\theta \neq 0$), so that the fluid motion is in the plane containing $\mathbf{k}$ and $\mathbf{B}_0$. Because U_{1x} is not required to be zero, (2.45) implies that η_1 is also nonzero; i.e., magnetoacoustic waves affect the plasma density and are thus called *compressional* waves. For magnetoacoustic waves,

$$\mathbf{v}_g = v_A \hat{\mathbf{k}}\,, \tag{2.52}$$

so that wave energy may flow at an arbitrary angle to $\hat{\mathbf{b}}$, as opposed to Alfvén waves (with $\mathbf{v}_g \parallel \hat{\mathbf{b}}$).

In a warm plasma, when β is no longer small relative to unity, the plasma pressure terms can no longer be ignored. The pressure gradient term is reinserted in (2.43) and the adiabatic equation of state (2.32) closes the set of equations. In this case a linear analysis yields a dispersion relation with three solutions:

$$\omega^2 = k^2 v_A^2 \cos^2\theta\,, \qquad \frac{\omega^2}{k^2} = \frac{1}{2}(v_A^2 + c_S^2) \pm \frac{1}{2}[(v_A^2 + c_S^2)^2 - 4 v_A^2 c_S^2 \cos^2\theta]^{\frac{1}{2}}\,, \tag{2.53}$$

with the sound speed $c_S = \sqrt{\Gamma p_0/\eta_0}$. These three solutions correspond to the Alfvén mode, and the fast (+) and slow (−) magnetoacoustic modes, so named because the phase speeds satisfy

$$v_{\text{fast}} \geq v_A \geq v_{\text{slow}}\,. \tag{2.54}$$

In the limit of small background magnetic field strengths, fast mode waves become gas sound waves, with the dispersion relation $\omega^2 = k^2 c_s^2$. In the cold plasma limit, fast mode waves become magnetoacoustic waves. In the small-field limit, slow mode waves become magnetoacoustic-like, with the dispersion relation $\omega^2 = k^2 v_A^2 \cos^2\theta$. These only have magnetoacoustic properties for small angles θ. In the cold plasma limit, slow mode waves (along the field lines) become gas-sound-like, with $\omega^2 = k^2 c_S^2 \cos^2\theta$.

Further Reading:

1. Sturrock, P. A., 1994, *Plasma Physics: An Introduction to the Theory of Astrophysical, Geophysical and Laboratory Plasmas*, Cambridge University Press, Cambridge, Chapters 11 & 12.

2. Kivelson, M. G. & Russell, C. T. (Eds), 1995, *Introduction to Space Physics*, Cambridge University Press, Cambridge, Chapters 2 & 11.

3. Cravens, T. E., 1997, *Physics of Solar System Plasmas*, Cambridge University Press, Cambridge, Chapter 4.