Exercises for Chapter 4

Plasma frequency

0. Show that the plasma oscillation frequency is given by \( \omega_p = \left( \omega_{pe}^2 + \omega_{pi}^2 \right)^{\frac{1}{2}} \)
where \( \omega_{pi} = \left( \frac{e^2 n_0}{\varepsilon_0 m_i} \right)^{\frac{1}{2}} \) if ion motion is taken into account. Hint: determine the equations of motion for an ion and electron slab displaced relative to one another. Which term in this expression is dominant? Why?

Waves in a plasma, no magnetic field

1. Plasma diagnostics
   Calculate \( n_e \) for the standard microwave frequencies used in the Plasma Physics Department; (i) 10 GHz, (ii) 35 GHz and (iii) 110 GHz.

2. Space Shuttle reentry
   The Space Shuttle communicates with the ground using three frequency bands; UHF 259.7 - 296.8 MHz, S-band 1.7 - 2.3 GHz and Ku-band 15.25 - 17.25 GHz. During reentry there is a communications blackout because of ionization of the air around the spacecraft. This lasts for about 15 minutes as the Shuttle descends from 80 km down to 50 km. Estimate the minimum plasma density.

Waves in a plasma, with magnetic field

3. Consider a plasma slab that has a density profile like:

   ![Density Profile Diagram]

   A wave, frequency \( \omega \), is launched into it from the outside. What is the maximum density that the plasma can have so that the wave will pass through
   (i) if \( B^0 = 0 \),
   (ii) if wave is LHCP along \( B^0 \),
   (iii) if wave is RHCP along \( B^0 \),
   (iv) if wave is O perpendicular to \( B^0 \),
   (v) (HARDER) if wave is X perpendicular to \( B^0 \)?

Waves in a plasma. Propagation along \( B^0 \)

4. Plasma diagnostics - Faraday Rotation
   A linearly-polarized laser beam enters a uniform plasma slab, thickness \( L \), parallel to the magnetic field.
   Obtain an expression for the Faraday Rotation of the beam in terms of \( B^0 \), \( n \) and \( L \).
   Assume \( \omega_{ce}, \omega_{pe} << \omega \) as it would be for a laser beam.
Waves in a plasma. Propagation perpendicular to $B^\theta$.

5. Plasma diagnostics

An $O$ wave microwave beam is transmitted through a plasma slab, thickness $L$, density $n << n_c$ everywhere.

Obtain expressions for the phase change of the wave across the slab
(i) if the density is uniform across the slab, and
(ii) (HARDER) if the density profile is a parabola.

The equation for the parabola is $n = n_{max} \left(1 - \frac{4x^2}{L^2} \right)$.

In this case express the phase change in terms of the average density.

6. Reflection of radio waves by the ionosphere

Suppose waves are being transmitted vertically upwards from a point on the earth where the earth’s magnetic field is horizontal. Take the value of the field to be $3 \times 10^{-5}$ T.

(i) Calculate the frequencies for reflections of $O$ waves from each layer.

(ii) Calculate frequencies of reflection of $X$ waves from the F2 layer.

You will need the following:

$O$ wave cutoff is given by $\omega = \omega_{pe}$ and

$X$ wave cutoff by $\omega^2 = \omega_{pe}^2 + \omega \omega_{ce}$.

Compare the values of $\omega_{ce}$ and $\omega_{pe}$. Does this allow you to simplify the latter expression?
(iii) Compare with an actual measurement.

The vertical axis is the time delay for the reflected signal but it has been relabelled effective height; the horizontal axis is frequency.

7. Electron cyclotron heating of a plasma

The waves will be launched from outside the plasma and be completely absorbed in the region where \( \omega \approx \omega_{ce} \). (When we allow heating at a harmonic of the \( \omega_{ce} \), take the velocities of the electrons and relativistic effects into account, the condition generalizes to \( \omega = \frac{n\omega_{ce}}{\gamma} + k|v_e| \).

Possible heating strategies can be checked using the CMA diagram.

For a tokamak plasma, \( B \approx \frac{1}{r} \), where \( r \) is the major radius. So the resonance will be in the region shown below. The field is higher on the inside and lower on the outside.

The magnetic field is in the toroidal direction.

Here are some possible strategies

(i) Launch \( O \) waves from the high-field side to fundamental resonance,
(ii) Launch \( O \) waves from the low-field side to fundamental resonance,
(iii) Launch \( X \) waves from the high-field side to fundamental resonance,
(iv) Launch \( X \) waves from the low-field side to fundamental resonance,
(v) Launch \( O \) waves from the high-field side to second harmonic,
(vi) Launch \( O \) waves from the low-field side to second harmonic,
(vii) Launch \( X \) waves from the high-field side to second harmonic,
(viii) Launch \( X \) waves from the low-field side to second harmonic.

Plot each of these on a CMA diagram and comment.
Each plot will start on the $\frac{\Omega_{pe}^2}{\Omega^2}$ axis and head towards the resonance. The variation in $B$ is small so there will be either the fundamental or second harmonic but not both. The first question is; can the resonance be reached?

(HARDER) Calculate the maximum value of $\frac{\Omega_{pe}^2}{\Omega_{ce}^2}$ in each case.

Note that (i) $n = 0$ outside the plasma, (ii) only one resonance, fundamental or second harmonic, will be present (Why?).

Waves in a plasma. Propagation at an arbitrary angle to $B^0$.

8. *Whistlers*

Whistlers are radio signals in the audio-frequency range that "whistle". A lightning stroke excites a pulse that travels from one hemisphere to another and back again. The wave travels in a duct of enhanced electron density that follows the magnetic field lines.

Simple theory.

Take the equation for propagation at an arbitrary angle and apply the *quasilongitudinal approximation*. $\theta$ is sufficiently small that

$$N^2 \simeq 1 - \frac{X}{1 \pm Y \cos \theta}$$

Above the F2 layer, $X > 1+Y$ so, since the whistler travels well above the ionosphere, only the minus sign is of interest.

In fact it is reasonable, particularly for low frequencies, to simplify this further to

$$N^2 \simeq \frac{X}{Y \cos \theta}.$$
Under these approximations,

(i) find $v_{ph}$
(ii) find $v_g$.
(iii) show delay time $\propto \frac{1}{\sqrt{f}}$.

(iv) (HARDER) The correct definition of group velocity is

$$v_g = \frac{\partial \omega}{\partial k_x} \hat{x} + \frac{\partial \omega}{\partial k_y} \hat{y} + \frac{\partial \omega}{\partial k_z} \hat{z}.$$ 

Apply this definition to the whistler dispersion equation to obtain $\theta_r$ the ray direction. Then by calculating the maximum value, show that this direction always lies within about 20° of the direction of $B^0$, i.e., the whistler wave is guided by the field line to within this angle.

When the quasilongitudinal approximation is used, nose whistlers are predicted.

Note that in laboratory situations, RHCP waves propagating where $\omega_{ce} > \omega$ are known as whistlers or helicon waves.

**Ion motions**

9. **Lower hybrid resonance**

The $X$ wave, with ion motions, has the dispersion relation

$$\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2 - \omega_{ce}\omega_{ci} + \frac{\omega^2(\omega_{ce} - \omega_{ci})^2}{\omega_p^2 - \omega^2 + \omega_{ce}\omega_{ci}}}$$

where $\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2$.

What is the resonant frequency when the density $n = 0$ and when $n = \infty$?

10. **Alfven waves**

(i) Show that when $\omega_{ce} < \omega_{pe}$, the dispersion relation for propagation along $B^0$ that includes ion motions yields $v_{ph} = c \sqrt{\frac{\omega_{ce}\omega_{ci}}{\omega_{pe}^2}}$ as $\omega \to 0$.

(ii) Define the Alfven speed $V_A = \frac{B}{\sqrt{\mu_0 \rho}}$. Show $v_{ph} = V_A$. 


Effects of geometry and boundaries

11. In a uniform cylindrical plasma with a conducting wall, the magnetic field components of the helicon wave are

\[\begin{align*}
B_z^I &= AJ^0_m(k_z r), \\
B_r^I &= jA \left( \frac{k_z m J^0_m(k_z r) + k^2 J^0_m(k_z r)}{k r} \right), \\
B_\theta^I &= -A \left( \frac{k_z m J^0_m(k_z r)}{k r} + \frac{k^2 J^0_m(k_z r)}{k r} \right)
\end{align*}\]

each with a factor \(e^{i(m \theta + k_z z - \omega t)}\).

Consider a cross-section of the plasma. Sketch the field lines for \(m = 0\) at different instants.

MHD waves

12. (i) Write down the first-order equations (in cartesian coordinates) corresponding to the cold plasma case where \(U = 0\),

\[\begin{align*}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{j} \\
\rho \frac{\partial \mathbf{v}}{\partial t} &= \mathbf{j} \times \mathbf{B} \\
0 &= \mathbf{E} + \mathbf{v} \times \mathbf{B},
\end{align*}\]

for the case where \(\mathbf{B}^0\) is in the \(z\)-direction, and \(\mathbf{k}\) has both \(x\) and \(z\) components.

(ii) Find the dispersion relations for all the waves. Are any missing?

13. Lower hybrid waves

See if there is a dispersion relation for electrostatic waves propagating perpendicular to \(\mathbf{B}^0\). Electrostatic waves means no wave magnetic field, or \(\nabla \times \mathbf{E} = 0\).

Use these equations

\[\begin{align*}
\frac{\partial \rho_e}{\partial t} + \nabla \cdot (\rho_e \mathbf{v}_e) &= 0 \\
\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{v}_i) &= 0 \\
\rho_e \frac{\partial \mathbf{v}_e}{\partial t} &= -n_e e (\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) \\
\rho_i \frac{\partial \mathbf{v}_i}{\partial t} &= n_i e (\mathbf{E} + \mathbf{v}_i \times \mathbf{B})
\end{align*}\]

and use \(n_e = n_i\).

In fact, using these equations, there are no waves. There are oscillations at the lower hybrid frequency \(\omega = \sqrt{\omega_e \omega_{ce}}\).
14. Solid-state plasmas

Similarities to gaseous plasmas
When plasma criteria are satisfied, can observe something resembling electrical discharges in gases, plasma confinement and waves.

Differences:
- density: below $10^{22} \text{ m}^{-3}$ in weakly-doped or intrinsic semiconductors up to $10^{29} \text{ m}^{-3}$ in metals.
- temperature: usually in equilibrium with the host lattice, room temperature down to liquid helium temperatures
- carrier masses: The effective masses of the electrons may be two orders of magnitude less than $m_e$; the effective masses of the holes are similar but not equal to those of the electrons. These masses depend on the direction with respect to the lattice.
- dielectric constant: $\varepsilon_r$ can be very large, e.g., $\approx 100$ for bismuth.

There are several kinds of solid-state plasma.

(1) **compensated**, where the numbers of mobile holes and mobile electrons are equal, i.e., $n_e = n_h$. Intrinsic semiconductors, semimetals (bismuth, antimony) and certain metals (iron, tungsten).

(2) **uncompensated**, $n_e >> n_h$ or $n_h >> n_e$ and overall charge neutrality is provided by the lattice of immobile ions. Doped or extrinsic semiconductors and other metals (sodium, copper).

(3) Apply an external electric field strong enough to cause avalanche breakdown.
(There are also liquid plasmas. For example, mercury, electrolytic solutions)

Waves in solid-state plasmas

The earlier equation for propagation along $B^0$ that included ion motions can be modified to

$$k^2 = \frac{\omega^2}{v^2} \left( \frac{\omega_{pe}^2}{\omega} \pm \frac{\omega_{ph}^2}{\omega} \pm \frac{\omega_{pi}^2}{\omega} \right)$$

where the subscripts $e$, $h$ and $i$ refer to electrons, holes and lattice ions (this expression assumes they are positive), respectively. The ion mass can be treated as infinite so the last term is zero. In a solid we should use $\varepsilon_r \varepsilon_0$ instead of $\varepsilon_0$ in calculating the plasma frequencies and the velocity of light in the material is $v = \frac{1}{\sqrt{\mu_0 \varepsilon_r \varepsilon_0}}$ instead of $c$ where $\varepsilon_r$ is the dielectric constant.

Rearranging gives

$$\frac{k^2 v^2}{\omega^2} = 1 \pm \left( \frac{\omega_{pe}^2}{\omega_{ce} (\omega \pm \omega_{ce})} - \frac{\omega_{ph}^2}{\omega_{ch} (\omega \pm \omega_{ch})} \right) \mp \left( \frac{\omega_{pe}^2}{\omega_{ce} \omega_{ce}} - \frac{\omega_{ph}^2}{\omega_{ch} \omega_{ch}} \right)$$
Alfven waves

(i) Show that if the plasma is compensated the last term vanishes. Further, if \( \omega_{ce or h} \gg \omega \) then the dispersion relation simplifies to

\[
\frac{1}{v_{ph}^2} = \frac{1}{v^2} + \frac{1}{V_A^2}.
\]

\( V_A \ll v \) so

\[ v_{ph} = V_A. \]

Damping is small if \( \omega \tau \gg 1 \).

An early experiment looked at the propagation of Alfven waves in a small 4 mm diameter cylinder of bismuth. The bismuth was cooled to liquid helium temperatures. The wave frequency was 16.25 GHz.

(ii) Calculate the Alfven velocity.

Use \( n = 3.1 \times 10^{23} \text{ m}^{-3} \), effective mass for electrons = 0.080 \( m_e \) (multiply this by 4.55 to take account of anisotropy), effective mass for holes = 0.068 \( m_e \) (multiply by 1), \( B = 1 \text{ T} \),

(iii) Check that \( \omega_{ce or h} \gg \omega \).

Helicon waves

If the plasma is uncompensated, the last term does not vanish. If it is n-type \( n_e \gg n_h \); if it is p-type \( n_h \gg n_e \). Again \( \omega_{ce or h} \gg \omega \). Only the lower sign gives a wave.

\[
v_{ph}^2 = v^2 \frac{\omega c e}{\omega p}.
\]

Damping is small if \( \omega c \tau \gg 1 \).

(iv) Compare this with the whistler-helicon dispersion relations discussed earlier.

(v) Show that you do not need to know the mass in this case.

One early experiment looked at helicon waves propagating along \( B^0 \) in the metal sodium. The sodium was cooled to liquid helium temperatures. They put their sample in a 1 T magnetic field and looked at standing waves on a length of 4 mm.

(vi) Predict the frequency of the lowest order. Use \( n = 2.7 \times 10^{28} \text{ m}^{-3} \).

Another early experiment looked at helicon waves propagating parallel to \( B^0 \) in the semiconductor indium antimonide using 9 GHz microwaves. The InSb sample was at room temperature. They looked at standing waves on a length of 2 mm.

(vii) Predict the magnetic field that gave the lowest frequency. Use \( n = 1.2 \times 10^{20} \text{ m}^{-3} \).