Lecture 5: Stellar structure

Senior Astrophysics

2018-03-14

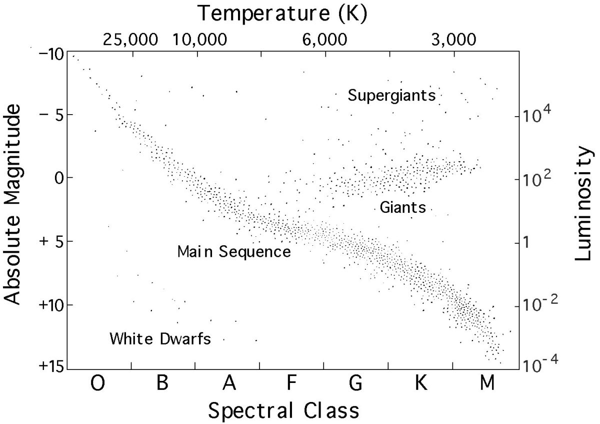
Outline

Part 2: Stars

Stellar structure and evolution (7 lectures + 2 labs)

1. How stars work
2. How stars evolve
3. Stellar remnants

Motivation



http://heasarc.gsfc.nasa.gov/docs/RXTE_Live/class.html][http://heasarc.gsfc.nasa.gov/docs/RXTE_Live/class.htmlhttp://heasarc

Stellar structure

What is a star?

- Held together by self-gravity
- Collapse is resisted by internal pressure
- Since stars continually radiate into space, there must be a continual energy source

Simplifying assumptions

- Stars are spherical and symmetrical
  all physical quantities depend only on r
- Ignore rotation
- Ignore outside gravitational influences
- Uniform initial composition
  no **initial** *dependence of composition on radius*
- Newtonian gravity
  no relativistic effects
- Stars change slowly with time
  can neglect d/dt *terms*

Stellar structure

To describe an isolated, static, symmetric star, we need four equations:

1. **Conservation of mass**
2. **Conservation of energy** (at each radius, the change in the energy flux equals the local rate of energy release)
3. **Equation of hydrostatic equilibrium** (at each radius, forces due to pressure differences balance gravity)
4. **Equation of energy transport** (relation between the energy flux and the local gradient of temperature)

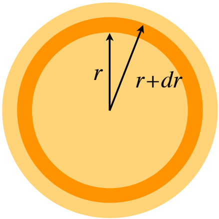
Stellar structure

In addition, we need to describe

1. **Equation of state** (pressure of the gas as a function of its density and temperature)
2. **Opacity** (how transparent the gas is to radiation)
3. **Nuclear energy generation rate** as $f(r, T)$

Conservation of mass

- If m is the mass interior to radius r, then m, r and ρ are not independent, because $m(r)$ is determined by $\rho(r)$.
- Consider a thin shell inside the star, radius r and thickness dr



Volume is $dV = 4\pi r^2 dr$, so mass of shell is

$$dm = 4\pi r^2 dr.\rho(r)$$

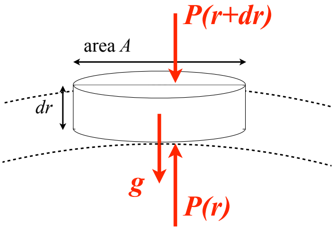
or

$$\frac{dm}{dr} = 4\pi r^2 \rho(r)$$

the equation of **mass conservation**

Hydrostatic equilibrium

- Consider a small parcel of gas at a distance r from the centre of the star, with density $\rho(r)$, area A and thickness dr.
- Outward force: pressure on bottom face $P(r)A$



- Inward force: pressure on top face, plus gravity due to material interior to r:

$$P(r{+}dr)A + \frac{Gm(r)dm}{r^2}$$
$$= P(r + dr)A + \frac{Gm(r)\rho A dr}{r^2}$$

Hydrostatic equilibrium

- In equilibrium forces balance, so

$$P(r)A = P(r+dr)A + \frac{Gm(r)\rho A dr}{r^2}$$

i.e.

$$\frac{P(r+dr)-P(r)}{dr} A\,dr = -\frac{Gm(r)}{r^2}\rho(r) A dr$$

or

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho$$

the equation of **hydrostatic equilibrium**

Estimate for central pressure

- We can use hydrostatic equilibrium to estimate P_c: we approximate the pressure gradient as a constant

$$\frac{dP}{dr} \sim -\frac{\Delta P}{\Delta R} = \frac{P_c}{R} = \frac{GM}{R^2}\rho$$

Now assume the star has constant density (!): so

$$\rho_c = \bar{\rho} = \frac{M}{V} \sim \frac{M}{\frac{4}{3}\pi R^3}$$

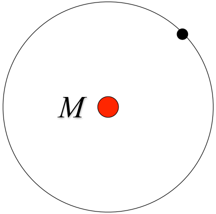
then so

$$P_c \sim \frac{3GM^2}{4\pi R^4}$$

For the Sun, we estimate $P_c \sim 3 \times 10^{14}\ \mathrm{N\,m^{-2}} = 3 \times 10^9$ atm.

Interlude: The virial theorem

- Gravity has a very important property which relates the gravitational energy of a star to its thermal energy.
- Consider a particle in a circular orbit of radius r around a mass M.



- Potential energy of particle is

$$\Omega = -\frac{GMm}{r}$$

- Velocity of particle is $v = \sqrt{\frac{GM}{r}}$ (Kepler)

The virial theorem

- So kinetic energy is
$$K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r}$$
i.e. $2K = -\Omega$ or $2K + \Omega = 0$.
- Total energy
$$\begin{aligned} E =& K + \Omega \\ =& \Omega - \frac{\Omega}{2} = \frac{\Omega}{2} < 0 \end{aligned}$$
- *Consequence*: when something loses energy in gravity it **speeds up**!

The virial theorem

- The virial theorem turns out to be true for a wide variety of systems, from clusters of galaxies to an ideal gas; thus for a star we also have
- $$\Omega + 2U = 0$$

  where U is the total internal (thermal) energy of the star and Ω is the total gravitational energy.
- So a *decrease* in total energy E leads to a decrease in Ω but an *increase* in U and hence T, i.e. when a star loses energy, it **heats up**.
- **Fundamental principle**: stars have a negative heat capacity: they heat up when their total energy decreases.
- This fact governs the fate of stars

Website of the Week:

NASA ADS
http://adsabs.harvard.edu/abstract_service.html
Querying the astronomical literature

Timescales

There are three important timescales in the life of stars:

1. dynamical timescale — *the time scale on which a star would expand or contract if the balance between pressure gradients and gravity was suddenly disrupted*
2. thermal timescale — *how long a star would take to radiate away its thermal energy if nuclear reactions stopped*
3. nuclear timescale — *how long a star would take to exhaust its nuclear fuel at the current rate*

Timescales

1. **Dynamical timescale**:
   - the timescale on which a star would expand or contract if it were not in equilibrium; also called the **free-fall timescale**

$$\tau_{\rm dyn} \equiv \frac{\text{characteristic radius}}{\text{characteristic velocity}} = \frac{R}{v_{\rm esc}}$$

Escape velocity from the surface of the star:

$$v_{\rm esc} = \sqrt{\frac{2GM}{R}}, \qquad \text{so}$$

$$\tau_{\rm dyn} = \sqrt{\frac{R^3}{2GM}}$$

For the sun, $\tau_{\rm dyn} \simeq 1100$ s

Timescales

1. **Thermal timescale**:
   - the timescale for the star to radiate away its energy if nuclear reactions were switched off: also called the **Kelvin-Helmhotz timescale**
   - Total gravitational energy available

$$E_{\rm grav} \sim \frac{GM^2}{R}$$

   - If the star radiates energy at L (J/s), then it can keep up this rate for

$$\tau_{\rm th} \sim \frac{E_{\rm grav}}{L} \sim \frac{GM^2}{RL}$$

For the sun, $\tau_{\rm th} \sim 3 \times 10^7$ y $\ll$ age of Earth.

Timescales

1. **Nuclear timescale**: times to exhaust nuclear fuel at current rate.

$$\tau_{\rm nuc} \sim \frac{\eta M_c c^2}{L}$$

where η is an efficiency factor for nuclear fusion: $\eta \sim 0.7\%$ (see next lecture), and M_c is the mass of the core.
For the sun, $\tau_{\rm nuc} \sim 10^{10}$ y

Timescales

- For stars,

$$\tau_{\rm dyn} \ll \tau_{\rm th} \ll \tau_{\rm nuc}$$

- $\tau_{\rm dyn}$ = timescale of collapsing star, e.g. supernova
- $\tau_{\rm th}$ = timescale of star before nuclear fusion starts, e.g. pre-main sequence lifetime
- $\tau_{\rm nuc}$ = timescale of star *during* nuclear fusion, i.e. main-sequence lifetime

Timescales

- Most stars, most of the time, are in hydrostatic and thermal equilibrium, with slow changes in structure and composition occurring on the (long) timescale τ_{nuc} as fusion occurs.
- If something happens to a star faster than one of these timescales, then it will NOT be in equilibrium.
  e.g. sudden addition of energy (nearby supernova?), sudden loss of mass (binary interactions)

Structure again

- We have derived two of the equations which define the structure of stars:

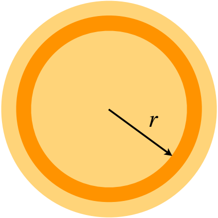
$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho \qquad \textbf{hydrostatic equilibrium}$$

$$\frac{dM}{dr} = 4\pi r^2\rho \qquad \textbf{mass conservation}$$

- We need two more equations

Equation of energy generation

- Assume the star is in thermal equilibrium, so at each radius T does not change with time.
- Rate of energy generation/unit mass $= \varepsilon$. Then



Shell mass $dm = 4\pi r^2 \rho dr$
Luminosity at r: $L(r)$
Luminosity at $r + dr$: $L(r) + dL$

$$dL = 4\pi r^2 \rho dr \times \varepsilon$$

so

$$\frac{dL}{dr} = 4\pi r^2 \rho \varepsilon$$

the equation of **energy generation**

Equation of energy transport

- Fourth equation describes how energy is transported through the layers of the star, i.e. how the gas affects the radiation as it travels through.
- Depends on local density, opacity and temperature gradient.
- Will not derive here, but quote result:

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2}$$

the equation of **energy transport**. a is called the **radiation constant** and has the value

$$a = \frac{4\sigma}{c} = 7.566 \times 10^{-16}\ \mathrm{Jm^{-3}K^{-4}}$$

where $\sigma = 5.670 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$ is the Stefan-Boltzmann constant. Comes from considering radiation pressure.

Equations of stellar structure

- This gives us four equations in four unknowns — $m(r)$, $L(r)$, $\rho(r)$ and $T(r)$ — so enough for a solution, provided we know $P(\rho, T)$, κ and ε.
- Also need **boundary conditions**:
  - centre of star ($r \to 0$): $M \to 0$, $L \to 0$
  - surface of star ($r \to R_*$): $T \to T_s$, $P \to 0$, $\rho \to 0$
- The calculation of full stellar models is a **very** hard problem, and must be done numerically, since in general κ and (especially) ε are strong functions of density and temperature.

Stellar models

- Some progress can be made by making simplifying assumptions, e.g. if pressure is **only** a function of density, then the first two equations can be solved independently from the equations involving temperature.
- We will be investigating numerical models of stars in Lab 2.

Next lecture

- is our first computer lab. This will be held in

SNH Learning Studio 4003

where we will be exploring the Saha-Boltzmann equation.

- Review your 2nd year Matlab notes, and perhaps bring them to the lab for reference.
- **Please read** the exercises before the lab. There's a lot in there (mostly repeat of lecture material), so don't want to lose time.
- If you can't make Friday's session, there's also a lab on **now** (10–11 am Wednesday).