# The fall and bounce of pencils and other elongated objects

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If an inclined pencil is released from rest with its bottom end resting on a table, the bottom end will slide forward or backward or it will remain at rest at the start of the fall, then slide backward for a short period before sliding forward. The magnitude and direction of the displacement of the bottom end of the pencil depends on the initial angle of inclination, the coefficient of friction, and on the length and mass distribution of the pencil. The same ground reaction forces play a similar role in the fall of trees and chimneys, the bounce of a football and any other elongated object, and in activities such as walking and running. When an elongated object is thrown obliquely to the ground, the object can bounce either forward or backward depending on the angle of inclination at impact. Spherical objects bounce away from the thrower. The difference arises because the horizontal friction force is determined not only by the normal reaction force, but also by the line of action of the normal reaction force relative to the center of mass. 
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## I. INTRODUCTION

Suppose that an inclined pencil is released from rest with its bottom end resting on a table. The subsequent motion of the pencil has been described for a frictionless table[1] and for a hinged stick[2] (the tip of which can accelerate faster than $g$ under certain conditions), but the result for a stick or pencil allowed to slide on a real table is different. On a frictionless table there is no horizontal friction force and no horizontal motion of the center of mass. The center of mass therefore falls vertically while the bottom end of the pencil slides backward. On a real table the bottom end of the pencil can slide either backward or forward depending on the initial angle of inclination and the coefficient of sliding friction $\mu$. In general, the bottom end slides backward at low initial angles of inclination to the horizontal and slides forward at high initial angles of inclination. Sometimes the bottom end will remain at rest at the start of the fall, then slide backward for a short period before sliding forward.

The falling pencil is just one of many examples where ground reaction forces control the dynamics. Other examples include walking and running,[3] the felling of a tree or chimney,[4,5] and the bounce of a football. An elongated object when dropped to the ground will have a falling or rotating phase while it remains in contact with the ground, with the result that the centripetal force on the object acts to reduce the normal reaction force. For example, the maximum walking speed of a person is limited by the fact that the person will become airborne if the rotation speed $v$ of his or her center of mass is large enough.[3] Given that the center of mass pivots about one foot on the ground, a walking stride will change to a running stride when $v^2/R > g$, where $R$ is the height of the center of mass. For most adults, $R$ is about 1 m so the maximum adult walking speed is about 3.1 $\mathrm{ms}^{-1}$. In this paper we consider only the case of a falling pencil because a pencil (or pen) will be within easy reach of all readers. However, the word "pencil" can be taken as a generic term for any elongated object, including a top-heavy human leg.

## II. EQUATIONS OF MOTION

The geometry is shown in Fig. 1 and the equations of motion are

$$F = M dv_x/dt \tag{1a}$$

$$N - Mg = M dv_y/dt \tag{1b}$$

and

$$I_{\mathrm{cm}} d\omega/dt = NH \sin\theta - FH \cos\theta, \tag{2}$$

where $M$ is the pencil mass, $v_x$ and $v_y$ are the velocity components of the center of mass, $H$ is the distance from the center of mass to the bottom end of the pencil, $I_{\mathrm{cm}}$ is the moment of inertia about an axis through the center of mass, and $\omega = d\theta/dt$. We will focus our attention on the relatively simple case where $v_x$, $v_y$, and $\omega$ are all zero and where $\theta = \theta_0$ at $t=0$. A qualitative discussion of the effects of a finite initial speed is given in Sec. V.

Simple observation indicates that $\omega$ increases with $t$ so from Eq. (2), $F < N \tan\theta$ during the fall. If the contact end slides on the table, then $F = \mu N$, which implies that $\tan\theta > \mu$. However, if $\tan\theta$ is less than $\mu$, $F$ will be less than $\mu N$, which means that the contact end grips the table instead of sliding. The pencil will then pivot about the contact end which will be prevented from sliding by a relatively small static friction force. There may be a difference between the static and sliding coefficients of friction but both coefficients are taken to be equal in this paper for simplicity. No qualitative differences result if the coefficients are different, but the algebra is more cumbersome.

If the pencil pivots about the contact point, then $v_x = H\omega \cos\theta$ and $v_y = -H\omega \sin\theta$, so

$$F = MH(\cos\theta\, d\omega/dt - \omega^2 \sin\theta) \tag{3}$$

and

$$N = Mg - MH(\sin\theta\, d\omega/dt + \omega^2 \cos\theta). \tag{4}$$

Equations (3) and (4) indicate that the acceleration of the center of mass has a component $H\omega^2$ directed toward the contact point and a component $d(H\omega)/dt$ directed at right angles to the pencil. The substitution of Eqs. (3) and (4) into Eq. (2) yields

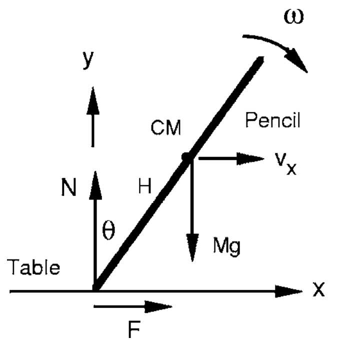


Fig. 1. Geometry of a falling pencil. The bottom end of the pencil is free to slide forward or backward on the table.

$$\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega_0^2 \sin\theta, \tag{5}$$

where $\omega_0^2 = MgH/I_0$, and $I_0 = I_{\text{cm}} + MH^2$ is the moment of inertia of the pencil about an axis through the contact end. Equation (5) can also be derived simply by equating the torque $MgH \sin\theta$ about the contact point to $I_0 d\omega/dt$. However, Eq. (5) remains valid only if the contact point remains at rest.[1] For small $\theta$ where $\sin\theta \approx \theta$, the solution of Eq. (5) is

$$\theta = \theta_0 (e^{\omega_0 t} + e^{-\omega_0 t})/2, \tag{6}$$

showing that the angular displacement and velocity grow exponentially rather than linearly with time, at least when $\omega_0 t$ is greater than about 3. Equation (5) can be integrated[2] to show that

$$\omega^2 = 2\omega_0^2 (\cos\theta_0 - \cos\theta), \tag{7}$$

which also follows immediately from energy conservation. $F$ and $N$ are therefore given by

$$F = MH\omega_0^2 \sin\theta (3\cos\theta - 2\cos\theta_0), \tag{8}$$

and

$$N = Mg - MH\omega_0^2 (1 + 2\cos\theta\cos\theta_0 - 3\cos^2\theta). \tag{9}$$

At time $t=0$ where $\theta=\theta_0$, $F=MH\omega_0^2 \sin\theta_0 \cos\theta_0$ and $N = Mg - MH\omega_0^2 \sin^2\theta_0$. If the pencil is released from rest from a near vertical position, then $F$ will be much smaller than $N$ at that time so the pencil will start to fall without sliding, as if the contact end were attached to a hinge. Equation (8) shows that for a near vertical launch where $\cos\theta_0 \approx 1$, $F$ increases with time to a maximum value of $0.305 MH\omega_0^2$ at $\theta = 26.7°$, decreases to zero at $\theta = 48.2°$, and then reverses sign. For the same launch condition, Eq. (9) indicates that $N$ decreases with time to the minimum value $Mg - 4MH\omega_0^2/3$ when $\cos\theta = 1/3$ or $\theta = 70.5°$.

If the pencil has a uniform mass distribution so that $I_0 = 4MH^2/3$, then the minimum value of $N$ is zero. Consequently, $N$ can decrease to a value where the magnitude of $F$ might be much larger than $\mu N$. However, sliding will commence if the magnitude of $F$ is equal to $\mu N$, in which case Eqs. (3) and (8) are no longer valid. Even if $F/N$ remains less than $\mu$ up to the time that $F$ reverses sign, the magnitude of $F/N$ will increase rapidly to $\mu$ after $F$ reverses sign. The pencil will therefore start to slide either forward or backward after the initial grip phase, depending on the magnitude of $\mu$. From Eqs. (8) and (9) it can be shown that the maximum positive value of $F/N$ is 0.371 when the pencil is released

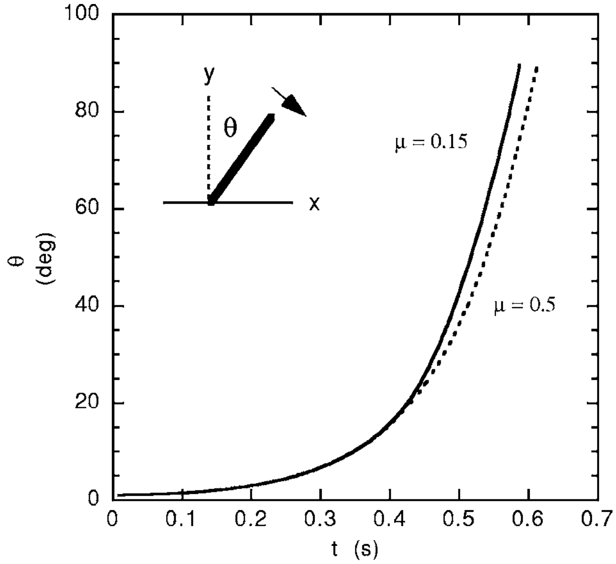


Fig. 2. The angle $\theta$ versus $t$ for two values of $\mu$ when $\theta_0 = 1°$, $M = 10$ g, and $H = 0.1$ m.

from a near vertical position and that $F/N$ is a maximum when $\theta = 34.9°$. Consequently, the bottom end of the pencil will start to slide backward if $\mu < 0.371$, but it will start to slide forward if $\mu > 0.371$.

During the sliding phase, Eq. (3) can be replaced by the relation $F = \mu N$, which yields

$$\frac{d^2\theta}{dt^2} = \frac{MH(g - H\omega^2\cos\theta)(\sin\theta - \mu\cos\theta)}{I_{\text{cm}} + MH^2 \sin\theta(\sin\theta - \mu\cos\theta)}. \tag{10}$$

If the sliding phase commences after $F$ reverses sign, then $F = -\mu N$, in which case the sign of $\mu$ in Eq. (10) needs to be reversed. Falling motion essentially terminates when $\theta = 90°$ at which point $d\omega/dt = MgH/I_0$ and $N = MgI_{\text{cm}}/I_0$. If the pencil falls onto the table at finite horizontal speed, a significant fraction of the observed total forward or backward displacement can occur as a result of sliding motion after the pencil has fallen into a horizontal position. In practice, sliding motion after the fall is usually also accompanied by bouncing of one or both ends of the pencil.

## III. NUMERICAL SOLUTIONS

Numerical solutions of Eqs. (5) and (10) are shown in Figs. 2, 3, and 4 for a pencil of mass 10 g with $H = 0.1$ m released from rest at $\theta_0 = 1°$ when $\mu = 0.15$ or $\mu = 0.5$. The pencil was assumed to have a uniform mass distribution with $I_0 = 4MH^2/3$. Also shown for comparison in Fig. 5 are solutions for a frictionless table where $\mu = 0$ and for a hinged pencil where $\mu$ is effectively infinite.

Comparison of Figs. 2 and 5 shows that the total fall time is relatively insensitive to $\mu$ unless $\mu$ is very small, the reason being that the grip phase at the beginning of the fall lasts for a relatively long time compared to the sliding phase. The fall time would be much longer than that shown in Fig. 2 if the release angle were significantly less than 1°, given that the angular displacement increases exponentially with time at the beginning of the fall.

Figure 3 shows the behavior of $F$ and $N$ during the fall. The behavior of $N$ is essentially the same as that for a hinged pencil in that $N$ drops almost to zero when $\theta = 70.5°$ and then

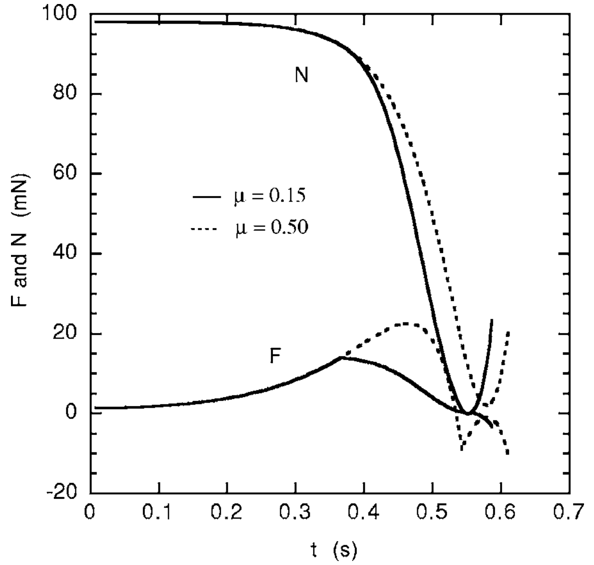


Fig. 3. $F$ and $N$ versus $t$ for the same conditions as those in Fig. 2. While $F<\mu N$, the pencil grips the table. When $F=\pm\mu N$, the pencil slides along the table.

increases slightly during the remainder of the fall. However, the behavior of $F$ is quite different after the pencil begins to slide. For a hinged pencil and the above parameters, $F$ increases to a maximum value of 22.5 mN and then decreases to $-147$ mN at the end of the fall. If the pencil is allowed to slide, it does so as soon as the magnitude of $F/N$ exceeds $\mu$, with the result that the pencil starts to slide backward if $\mu<0.371$ or slides forward if $\mu>0.371$.

The displacement of the bottom end of the pencil is shown in Fig. 4. If $\mu<0.371$, the initial backward slide gives way to a forward slide when $F$ reverses direction. At the beginning of the fall $F$ must be positive in order for the center of mass to accelerate forward. Toward the end of the fall where $\omega$ is large and where the forward acceleration of the center of mass comes to an end, $F$ changes sign because the horizontal

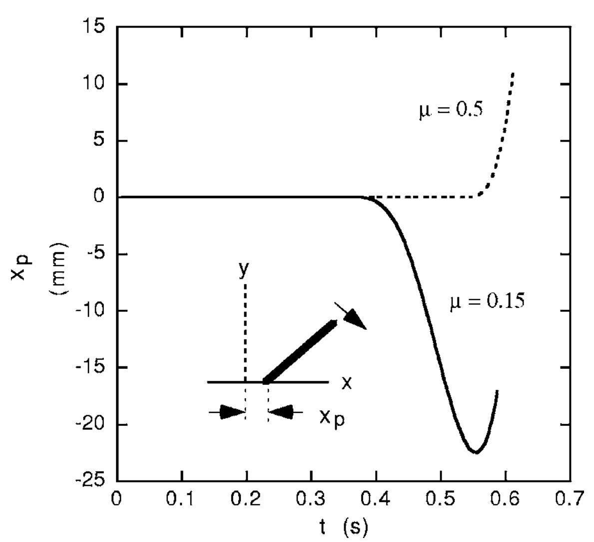


Fig. 4. The horizontal displacement of the bottom end of the pencil from its initial position $x_p$ versus $t$ (same conditions as Fig. 2). While the pencil grips the table $x_p$ remains zero.

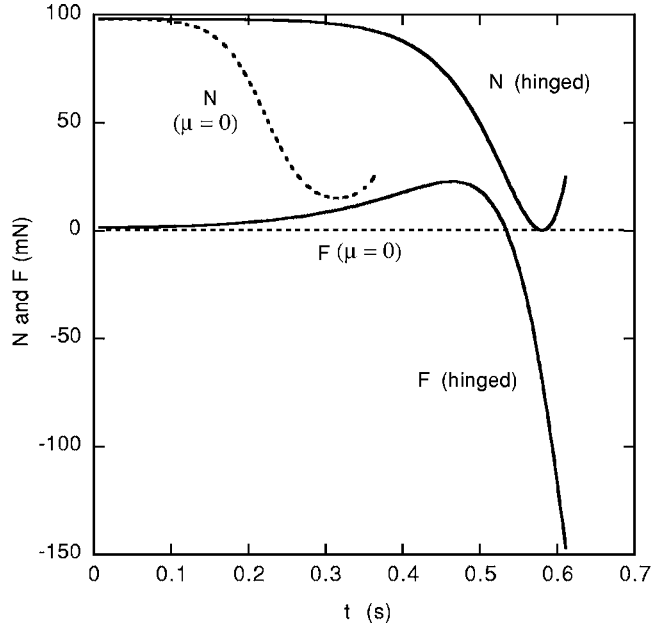


Fig. 5. $N$ and $F$ versus $t$ for a 10 g pencil with $H=0.1$ m released from rest at $\theta_0=1°$ when the bottom end slides on a frictionless table or when the bottom end is hinged.

component of the centripetal acceleration exceeds the horizontal component of the linear acceleration, as indicated by Eq. (3).

Solutions for other values of $M$ and $H$ show that the fall time is independent of $M$ and is directly proportional to $H^{1/2}$. Similarly, $\omega$ is independent of $M$ and is inversely proportional to $H^{1/2}$. This result is expected from Eqs. (5) and (6), given that $\omega_0$ is independent of $M$ and inversely proportional to $H^{1/2}$. The net forward or backward displacement of the bottom end of the pencil was found to increase linearly with $H$. For demonstration or experimental purposes, the displacement would therefore be easier to observe visually and better time resolution could be obtained on film using a meter stick rather than a pencil.

Under free fall conditions, the center of mass of a pencil released from rest will drop through a height $H$ in time $t=(2H/g)^{1/2}$ or in 0.143 s when $H=0.1$ m. If a 20 cm pencil is released at an angle of 1° from the vertical when resting on a table, the pencil takes about 0.6 s to fall onto the table. The standard "faster than $g$" falling stick demonstration works only if $\theta_0>42.1°$.[2]

## IV. IS LIFTOFF POSSIBLE DURING THE FALL?

A surprising result of the above calculations is that $N$ can drop to a minimum value of zero (without reversing sign), at least for a pencil with a uniform mass distribution. Such a result implies that the center of mass of the pencil can fall with a maximum acceleration equal to $g$ but it cannot fall any faster. However, Eq. (9) indicates that $N$ can have a minimum value less than zero if $I_0<4MH^2/3$, a situation that can arise if the pencil is top heavy. Suppose that a mass $m$ is attached to the top end of a pencil of length $L$ so that $I_0=(M/3+m)L^2$ and $H=0.5(M+2m)L/(M+m)$. If we assume that the pencil is released from a near vertical position and pivots about a fixed point at the bottom end, then the minimum value of $N$ according to Eq. (9) is $-m^2g/(M+3m)$.

From Eq. (1) a negative value of $N$ implies that the magnitude of $dv_y/dt$ is larger than $g$. $N$ will be negative near $\theta = 70°$ if the pencil is hinged at the bottom end, suggesting that the bottom end is likely to lift off the table if it is not anchored down. However, a pencil that is free to slide on a table will do so before $N$ drops to zero and before $N$ has a chance to reverse sign, because the ratio $F/N$ approaches infinity as $N$ approaches zero.

Suppose that the bottom end of a top-heavy, sliding pencil lifts off the table when $N$ drops to zero. $N$ will remain zero after liftoff, $F$ will also drop to zero, and hence $\omega$ will remain constant at its liftoff value. The bottom end will then rise vertically at speed $H\omega \sin\theta$ with respect to the center of mass, while the center of mass will fall with $dv_y/dt = -g$. Because $\theta$ continues to increase with time, the bottom end will rise at an increasing velocity with respect to the center of mass but the center of mass falls with increasing velocity after liftoff. In order to remain airborne, the pencil would need to rotate at a relatively high speed at liftoff. For example, if $h$ is about 0.15 m and if the pencil lifts off the table when $\theta$ is about 50°, a simple calculation indicates that $\omega$ would need to exceed about 70 rad/s in order for the pencil to remain airborne. In fact, the angular velocity of such a pencil released from rest is typically less than about 10 rad/s. Liftoff is therefore not possible under these conditions, as suspected by Theron.[2] Nevertheless, liftoff would be possible if the pencil were launched with a high initial angular velocity. An example of this effect is provided by an oval-shaped football that rolls end over end. Even at moderately low speeds the football becomes airborne because the condition $v^2/R > g$ is satisfied at relatively small $v$ when $R$ is small.

## V. PENCIL THROWN AT FINITE SPEED

While experimenting with falling pencils I noticed another interesting effect, presumably discovered previously by many others. That is, if the blunt end of a pencil is speared obliquely onto a horizontal surface, then the pencil can bounce straight back to the thrower. The effect is easier to observe and easier to control if the blunt end contains an eraser, because the pencil bounces to a greater height and because it more readily grips the surface. The backward bounce effect is intriguing because a spherical ball almost always bounces forward when thrown obliquely onto a surface. The exception to this general rule is that a spherical ball can bounce backward if it is projected with backspin at an angle close to the normal. However, an elongated object bounces backward even without backspin, provided it is inclined at a backward angle when it impacts the surface.

The essential difference between the bounce of spherical and nonspherical objects is that the normal reaction force usually acts through the center of mass of a spherical ball, but it can act well ahead of or well behind the center of mass of an elongated object, depending on the angle of inclination at impact. As a result, the net torque about the center of mass of an elongated object can be significantly larger or smaller than that on a spherical object of similar mass. The torque on an object determines the rate at which it spins, but why should it affect the change in horizontal velocity?

To compare the bounce of spherical and nonspherical objects, it is instructive to compare the bounce of a basketball with that of an oval shaped football of the same mass and stiffness. If both are thrown without spin at the same speed and angle of incidence, both will be subjected to the same normal reaction force and the same sliding friction force and both will remain in contact with the ground for about 20 ms. Consequently, we might expect that both balls should bounce to a similar height and both should slow down by the same amount in the horizontal direction. In fact, the basketball will bounce forward, but the football can bounce either forward or backward depending on the angle of inclination of the long axis at impact, as indicated in Fig. 6.

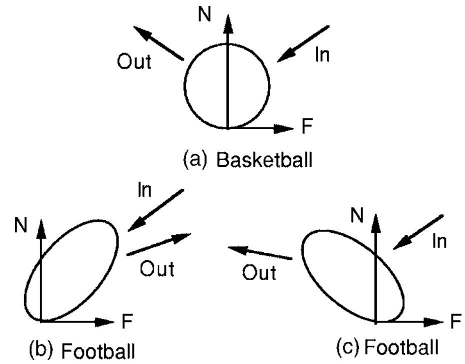


Fig. 6. A basketball and a football of the same mass and stiffness, incident without spin at the same speed, and angle of incidence on the same surface, bounce in different directions. The bounce direction of a football depends on its initial angle of inclination. The normal reaction force $N$ acts through the center of mass in (a), ahead of the center of mass in (b), and behind the center of mass in (c).

Both balls will initially slide along the surface and the friction force will act in a backward direction to reduce the horizontal speed of each ball. Sliding friction can act to bring an object to rest but it cannot reverse the direction of motion. Consequently, the backward bounce of a football must be associated with a static friction force. The friction force on a spherical ball causes the ball to rotate forward as it slides until the contact region at the bottom of the ball comes to rest. The contact region comes to rest when the rotation speed of the ball $R\omega$ is equal to the horizontal speed of the center of mass, where $R$ is the ball radius. The contact region of the ball will then grip the surface due to the large normal reaction force acting on the ball while it is compressed. The result is that the static friction force on the ball decreases to zero and then reverses direction, effectively propelling the ball forward.[6] Even though the bottom of the ball comes to rest during the grip phase, the center of mass of the ball continues to move forward throughout the entire 20 ms bounce period. The static friction force is determined by the extent of the elastic deformation of the ball in the horizontal direction, in the same way that the normal reaction force is determined by elastic deformation in the vertical direction.

When a football impacts the ground at a backward angle, the torque due to the friction force is opposed by the torque due to the normal reaction force, as indicated in Fig. 6(b). Consequently, the ball can slide forward without significant rotation, in which case the same sliding friction force acts on the ball as it does on the basketball but it acts for a longer time interval. If there is no rotation of the ball while it slides, then the whole ball would come to rest when the contact region comes to rest, provided the ball is infinitely rigid. In fact, a football is relatively flexible and will experience elastic deformation in the horizontal direction as a result of the shear force at the bottom of the ball. At the instant when the bottom of the ball comes to rest, the upper part of the ball will still be moving forward and it will continue to do so until the whole ball comes to rest in the horizontal direction. At that point the ball might still be compressing vertically and it will also store elastic energy due to its horizontal deformation. While the ball remains stretched in the horizontal direction, it exerts a forward force on the ground so the friction force continues to act in the backward direction and propels the ball backward. The details of this process are complicated by the fact that the friction force reverses direction several times while the ball grips the surface, both on a basketball and on a football. Multiple reversals of the friction force are due to transverse vibrations of the ball that occur while the ball undergoes one half of an oscillation in the vertical direction. Nevertheless, the net horizontal impulse acting on a football that bounces backward is clearly greater than that on a forward bouncing basketball with the same initial horizontal momentum. Measurements of the friction force acting on a basketball have been reported.[6] Measurements of the unusual bounce properties and the friction force acting on a football will be described in a separate publication.

## VI. CONCLUSIONS

The behavior of a falling pencil can be explained formally in terms of the relevant dynamic equations. The behavior also has a simple qualitative explanation that could illustrate the nature of accelerated circular motion for teaching purposes. That is, the horizontal force at the bottom end must be initially positive in order for the center of mass to move forward. The initial horizontal acceleration is relatively small and the required force is supplied by static friction. As the pencil accelerates, the bottom end retains its grip if the coefficient of static friction is large enough, but the bottom end will commence to slide backward if the coefficient of static friction is small. Near the end of the fall the friction force reverses direction because the centripetal force is then larger than that required to accelerate the center of mass forward.

The backward bounce of a pencil that is speared obliquely onto a horizontal surface illustrates an effect that is probably not well known. That is, the magnitude and direction of the friction force on an object depends not only on the normal reaction force but also on the line of action of the normal reaction force relative to the center of mass. The line of action determines the rate of rotation and hence determines the duration of the sliding period before the contact region comes to rest. While the contact region remains at rest, the magnitude and direction of the static friction force depends on the extent and direction of elastic deformation in a direction parallel to the surface. It also depends on the separate periods of oscillation of the object in directions parallel and perpendicular to the surface.

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