Chapter 1

Orbit Theory and Particle Drifts

1.1 Outline of Lecture 1

Aim: To develop a detailed knowledge of particle motion in homogeneous electric and inhomogenous magnetic fields. This is required to understand plasma motions, the development of magnetospheric structure and the solar wind, particle acceleration at shocks, and instabilities that produce plasma waves, all of which will be studied in future lectures.

Expected Learning Outcomes: Students are expected to be able to

- understand particle motions in terms of the parallel motion and transverse drifts perpendicular to $\mathbf{B}$ of the gyrocenter, and the gyromotion;
- show why particle energization can occur due to the combination of electric fields and plasma drifts,
- derive and explain quantitatively the physics of the $\mathbf{E} \times \mathbf{B}$ drift,
- explain the physics of the ∇B and curvature drifts and use the associated formulae,
- describe what adiabatic invariants are and why they are important,
- use conservation of the first adiabatic invariant (the magnetic moment) to describe particle motion in a magnetic bottle/mirror.

The lecture is ordered as follows:

1. General Considerations.
2. Motion with $E_{\parallel} \neq 0$ and homogeneous $\mathbf{B}$.
3. $\mathbf{E} \times \mathbf{B}$ and $\mathbf{F}_{\perp} \times \mathbf{B}$ drifts. .
4. Drifts in inhomogeneous magnetic fields.
5. Adiabatic invariants.
6. Magnetic mirror physics.

1.2 General Considerations

We start with motions of an individual, charged plasma particle subject to imposed, external electric, magnetic and other force fields. This is obviously simpler than studying motions of finite volumes of plasma since the electromagnetic and collisional interactions between charged plasma particles are ignored. Collective (wave) effects are also ignored here. Nevertheless, since many plasmas are collisionless and since a plasma's internal electromagnetic interactions are often unimportant compared with macroscopic fields (e.g., Earth's magnetic field), orbit theory often describes the motion of the plasma as a whole. An example of this is the flow of the solar wind plasma across its magnetic field.

Orbit theory is also important in understanding the motion of energetic particles, which often act as test particles, and in understanding the acceleration of particles. Examples of the former are the motion of energetic particles in the ring current and Van Allen radiation belts in Earth's inner magnetosphere, while the latter is exemplified in drift acceleration at shock waves. Orbit theory is also important in understanding the creation of particle distributions with free energy for wave growth, for instance in Earth's foreshock.

The basic equation for orbit theory is the (non-relativistic) equation of motion

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.1}$$

for a particle with mass m and charge q moving in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. This can be generalized by including additional forces such as gravity. The total derivative in equation (1.1) can be separated into

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}.\nabla \ . \tag{1.2}$$

One basic technique in orbit theory is to write the particle velocity as the sum

$$\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_D + \mathbf{v}_g \tag{1.3}$$

of three terms. The first, $\mathbf{v}_{\|}$, is the particle velocity parallel to the magnetic field, otherwise known as the particle's **parallel velocity**. The second, $\mathbf{v}_D$, is the **drift velocity of the particle's gyrocenter perpendicular to the magnetic field**: this drift velocity is associated with electric or other forces directed perpendicular to the magnetic field or else temporal or spatial variations in electric or magnetic fields. The sum $\mathbf{v}_{\|} + \mathbf{v}_D$ describes the velocity of the particle's **gyrocenter**. The final component, $\mathbf{v}_g$, is the particle's intrinsic **gyromotion** or cyclotron motion about its gyrocenter (and the magnetic field).

From Eq. (1.1) the time rate of change of a particle's kinetic energy is

$$\frac{d}{dt}(1/2mv^2) = \mathbf{v}.m\frac{d\mathbf{v}}{dt} = q\mathbf{v}.\mathbf{E} \ . \tag{1.4}$$

Energization of a particle therefore requires, as expected, the existence of a non-zero electric field such that at least one component of the particle velocity produces a non-zero value of $\mathbf{v}.\mathbf{E}$. Parallel electric fields can therefore energize particles. However, from Eq. (1.3) it can be seen that drifts $\mathbf{v}_D$ can also lead to particle energization.

The analyses below assume time-stationary macroscopic fields $\mathbf{E}$ and $\mathbf{B}$ unless otherwise stated. Time-varying electric and magnetic fields are sometimes important, however. For instance, time-varying electric fields lead to so-called "betatron" acceleration. Moreover, the time-varying fields of plasma waves can accelerate charged particles.

1.3 Motion with $E_\parallel \neq 0$ and homogeneous B.

Consider time-stationary plasmas with $\mathbf{E} = \mathbf{E}_\parallel \neq 0$ and a homogeneous magnetic field $\mathbf{B}$. Then the parallel equation of motion becomes

$$m\frac{dv_\parallel}{dt} = qE_\parallel \ , \tag{1.5}$$

which has the obvious solution $v_\parallel(t) = v_\parallel(0) + qE_\parallel t/m$.

Importantly the motions parallel and perpendicular to the magnetic field are, in general, separable. The velocity perpendicular to the magnetic field, $\mathbf{v}_\perp$, (with $\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp$ and $\mathbf{v}_\perp.\mathbf{B} = 0$) obeys the equation

$$m\frac{d\mathbf{v}_\perp}{dt} = q\mathbf{v}_\perp \times \mathbf{B} \ . \tag{1.6}$$

Differentiating Eq. (1.6) with respect to time and using Eq. (1.1) it is easily shown that

$$m\frac{d^2\mathbf{v}_\perp}{d^2t} = -\frac{q^2B^2}{m}\mathbf{v}_\perp = -m\Omega_c^2\mathbf{v}_\perp \ . \tag{1.7}$$

This equation shows simple harmonic motion; the quantity

$$\Omega_c = \frac{qB}{m} \tag{1.8}$$

is the (angular) gyrofrequency or cyclotron frequency of the particle. The gyrofrequency depends on B and the charge and mass of the particle. The gyromotion itself can be constructed by noting that the particle's acceleration is perpendicular to $\mathbf{B}$ and $\mathbf{v}_\perp$ (Eq. 1.6), and that the sense of rotation depends on the charge.

The **gyroperiod** T_c is the time for a particle to complete one cyclotron orbit:

$$T = \frac{2\pi}{\Omega_c} \ . \tag{1.9}$$

Note that the electron gyroperiod is ~ 2000 times shorter than the proton gyroperiod.

The **gyroradius** r_L (or Larmor radius) is the radius of a particle's circular motion about a magnetic field line. By integrating Eq. (1.7) it can be shown that

$$r_L = mv_\perp/qB = v_\perp/\Omega_c \ . \tag{1.10}$$

Moreover it can be shown that the sense of a particle's gyromotion relative to the magnetic field direction depends on the particle's charge, either using Eq. (1.6) or directly using Eq. (1.1): protons gyrate in a clockwise sense and electrons in an anti-clockwise sense. (Figure 1.1)

Consider next the current and magnetic field associated with charged particles gyrating about the magnetic field. Inspection quickly shows that these fields are anti-parallel to the background magnetic field $\mathbf{B}$. Accordingly, plasma particles are **diamagnetic**.

Exercise 1.1: Construct the gyromotion of a particle in coordinate space and show that the definition (1.10) for r_L is correct.

Exercise 1.2: Demonstrate that Figure 1.1 is correct, with protons and electrons gyrating in opposite screw senses relative to the magnetic field direction, and that plasma particles are diamagnetic.

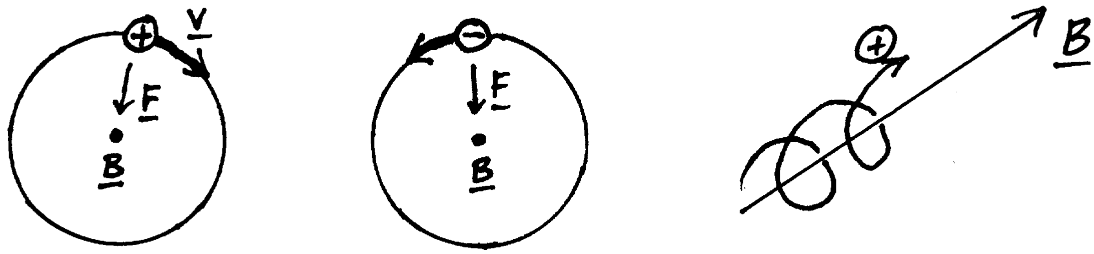

Figure 1.1: Figures showing that positively charged particles gyrate clockwise relative to **B** while negatively charged particles gyrate anti-clockwise.

1.4 Motion in static, homogeneous situations with B $\neq 0$ and other forces

This subsection treats particles moving in a time-invariant and homogeneous plasma subject to magnetic and other forces **F**. These other forces include gravity and electric forces. The parallel and perpendicular motions can be split as before:

$$m\frac{dv_\parallel}{dt} = F_\parallel \ , \tag{1.11}$$

$$m\frac{d\mathbf{v}_\perp}{dt} = \mathbf{F}_\perp + q\mathbf{v}_\perp \times \mathbf{B} \ . \tag{1.12}$$

The parallel motion has the obvious solution. For the perpendicular motion we assume the form (1.3), i.e., $\mathbf{v}_\perp = \mathbf{v}_D + \mathbf{v}_g$. Substituting in and rearranging leads to

$$m\frac{d\mathbf{v}_D}{dt} + m\frac{d\mathbf{v}_g}{dt} = (\mathbf{F}_\perp + q\mathbf{v}_D \times \mathbf{B}) + q\mathbf{v}_g \times \mathbf{B} \ . \tag{1.13}$$

Cancelling out the terms corresponding to the usual gyromotion, then the solution for a time-invariant drift is given by

$$\mathbf{F}_\perp = -q\mathbf{v}_D \times \mathbf{B} \ . \tag{1.14}$$

Requiring that $\mathbf{v}_D$ be perpendicular to **B** then leads to the solution

$$\mathbf{v}_D = \frac{1}{q}\,\frac{\mathbf{F} \times \mathbf{B}}{B^2} \ . \tag{1.15}$$

That is, particles subject to a force with a component perpendicular to the magnetic field will undergo a steady drift perpendicular to both the magnetic field and the perpendicular component of the force. Figure 1.2 shows that this can be understood physically in terms of the force increasing (decreasing) $v_\perp$ at the top (bottom) of the orbit relative to the direction of the force, thereby increasing (decreasing) r_L and so the length of the orbit perpendicular to both **B** and $\mathbf{F}_\perp$, and leading to a net drift of the particle in the direction given by Eq. (1.15).

The most common application of (1.15) is when the force is provided by a perpendicular electric field **E**. Since $\mathbf{F}_\perp = q\mathbf{E}_\perp$ then, the so-called $E \times B$ **drift velocity** is then

$$\mathbf{v}_{E\times B} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \ . \tag{1.16}$$

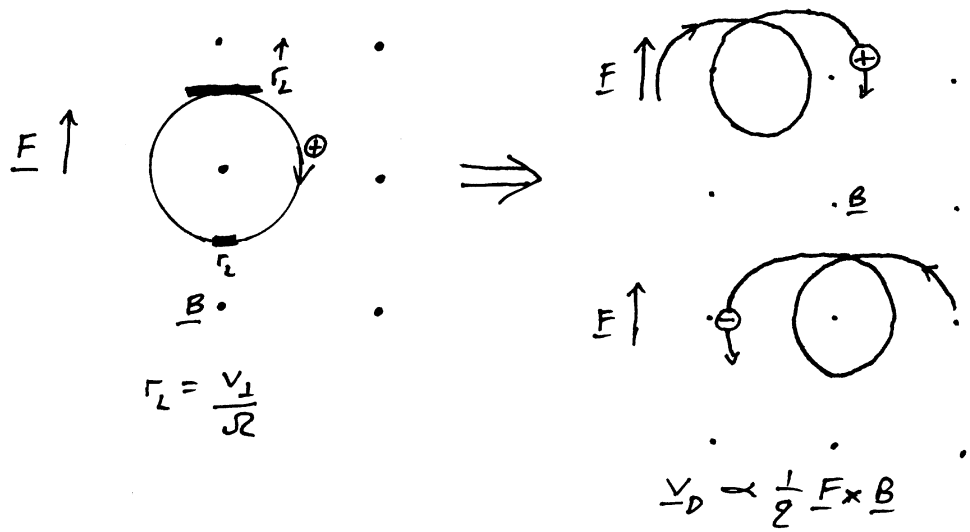

Figure 1.2: Drift motions due to an arbitrary force **F** perpendicular to the magnetic field.

Very importantly, the $E \times B$ drift velocity is independent of the particle charge. This means that plasma may undergo bulk motion due to an $E \times B$ drift, with no charge separation or build-up of ambipolar electric fields due to particles with different charges or energies moving with different drift velocities.

The solar wind provides a specific illustration of this: usually the solar wind velocity $\mathbf{v}_{sw}$ is not parallel to the magnetic field $\mathbf{B}_{sw}$, and it may be asked how the plasma can maintain itself in this state. The way it does this is by setting up and maintaining a "convection electric field" $\mathbf{E}_{sw} = -\mathbf{v}_{sw} \times \mathbf{B}_{sw}$ in the plasma. Then the component of the solar wind's velocity perpendicular to the magnetic field is just $\mathbf{v}_{sw,\perp} = \mathbf{E}_{sw} \times \mathbf{B}_{sw}/B^2$. The motion of an individual solar wind plasma particle is thus made up of a speed parallel to $\mathbf{B}_{sw}$, the $E \times B$ drift velocity and the gyromotion.

Exercise 1.3: Show that the situation of a magnetic field perpendicular to a gravitational field $\mathbf{g}$ leads to a plasma drift with velocity $\mathbf{v}_D = \mathbf{g} \times \mathbf{B}/qB^2$ that is mass independent but dependent on charge. What are the possible consequences of this charge dependence?

1.5 Motion in non-uniform magnetic fields

Often the magnetic field in a plasma varies with position or time, causing the plasma particles to drift, change their perpendicular kinetic energy, and sometimes to be energized as a result. First consider the effects of a gradient in magnetic field strength that is perpendicular to **B**.

1.5.1 ∇B drift, with $\nabla B \perp B$

Figure 1.3 shows the path of a positively charged particle in this case. Assuming that the gradient is on scale lengths long compared with the gyroradius, i.e.,

$$r_L \ll L = \left(|\nabla B|/|\mathbf{B}|\right)^{-1} \ , \tag{1.17}$$

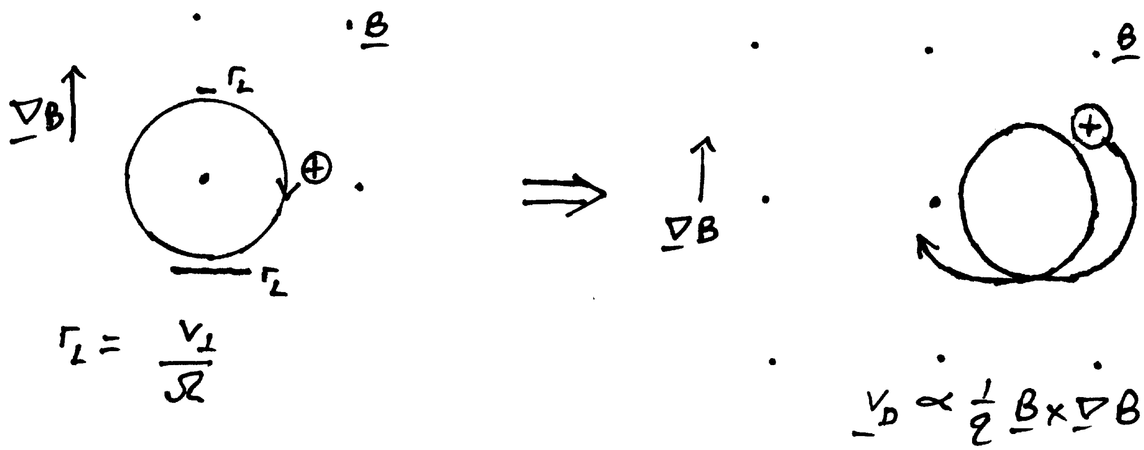

Figure 1.3: Drift velocity $\mathbf{v}_{\nabla B}$ due to a gradient ∇B in the strength of the magnetic field $\mathbf{B}$.

the orbit is almost circular but does not quite close. Since $r_L \propto B^{-1}$, r_L will be slightly smaller where B is larger and slightly larger where B is smaller. This causes the orbit to drift in the direction of its gyromotion where r_L is larger, perpendicular to both ∇B and to $\mathbf{B}$. A negatively charged particle drifts in the opposite direction. The physics is thus clear.

Quantitatively we write

$$\mathbf{B}(\mathbf{r} = \mathbf{x}_c) = \mathbf{b}\left(B(0) + \mathbf{x}_c.\nabla B(0)\right) \ , \tag{1.18}$$

where $\mathbf{x}_c$ represents the unperturbed gyromotion and $\mathbf{b}$ is the unit vector of the magnetic field. The perpendicular equation of motion is

$$\frac{d\mathbf{v}_\perp}{dt} = \Omega_c \mathbf{v}_\perp \times \mathbf{b}\left(1 + \frac{\mathbf{x}_c.\nabla B}{B(0)}\right) \ . \tag{1.19}$$

The particle velocity is now written as $\mathbf{v}_\perp = \mathbf{v}_c + \mathbf{v}_{\perp,1}$, where $\mathbf{v}_c$ represents the unperturbed gyromotion and $\mathbf{v}_{\perp,1}$ is the sum of the drift velocity and any first order perturbations to the gyromotion. Substituting into (1.19), grouping the zeroth order terms and deleting them, and ignoring the second order term $\mathbf{v}_{\perp,1} \times \mathbf{b}\mathbf{x}_c.\nabla B(0)$, the first order equation becomes

$$\frac{d\mathbf{v}_{\perp,1}}{dt} = \Omega_c\left(\mathbf{v}_c \times \mathbf{b}\mathbf{x}_c.\nabla B/B_(0) + \mathbf{v}_{\perp,1} \times \mathbf{b}\right) \ . \tag{1.20}$$

This equation is next averaged over a gyroperiod and $\mathbf{v}_{\perp,1}$ is identified as the constant drift velocity $\mathbf{v}_D$, so that the time derivative becomes zero and the drift velocity obeys the equation

$$\mathbf{v}_D \times \mathbf{b} = - < \mathbf{v}_c \times \mathbf{b}\, \frac{\mathbf{x}_c.\nabla B}{B(0)} > \ . \tag{1.21}$$

Now $\mathbf{v}_c = -\mathbf{x}_c \times \mathbf{b}\Omega_c$ and the time-average of the righthand term simplifies considerably since the x component of the term $< \mathbf{x}_c \mathbf{x}_c.\nabla B >$becomes

$$< x_{c,x}\mathbf{x}_c.\nabla B > \quad = \quad < x_{c,x}(x_{c,x}\left(\frac{\partial B}{\partial x} + x_{c,y}\frac{\partial B}{\partial y}\right) > \tag{1.22}$$

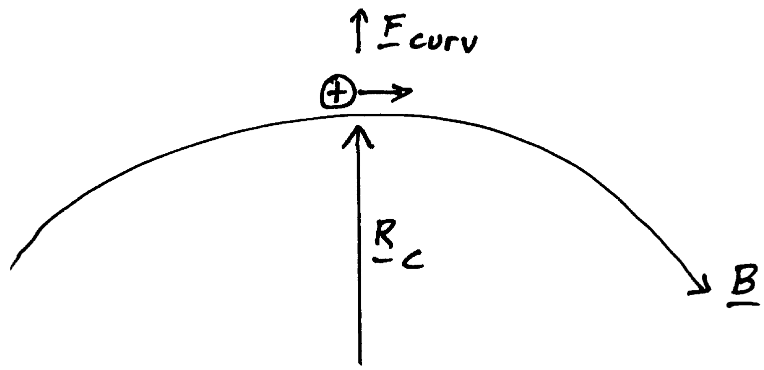

Figure 1.4: Definition of the radius of curvature $\mathbf{R}_c$ of a curved magnetic field line and the centrifrugal force $\mathbf{F}_{curv}$ experienced by a particle moving along this field line.

$$\begin{aligned} &= < x_{c,x}^2 > \frac{\partial B}{\partial x} \\ &= r_L^2/2 \frac{\partial B}{\partial x} \ , \end{aligned} \tag{1.23}$$

with a similar result for the y component. That is,

$$\mathbf{V}_D \times \mathbf{b} = \frac{r_L^2 \Omega_c}{2} \ \frac{\nabla B}{B} \ . \tag{1.24}$$

Rearranging, the final result is

$$\mathbf{v}_{\nabla B} = \frac{1}{2} \frac{m v_\perp^2}{qB} \frac{\mathbf{B} \times \nabla B}{B^2} \tag{1.25}$$

$$= \frac{1}{2} v_\perp r_L \frac{\mathbf{B} \times \nabla B}{B^2} \ . \tag{1.26}$$

The ∇B drift speed therefore depends on the charge, mass, and perpendicular energy of the particle, as well as on the magnetic field strength and the scale length of the gradient. This drift can therefore cause currents and charge separations in the plasma. Moreover, as seen below, the combination of a convection electric field and a ∇B plasma drift can lead to particle acceleration.

1.5.2 Curvature drifts

Curvature of magnetic field lines can also cause plasma particles to drift. Figure 1.4 shows this situation. As the particle moves along a curved magnetic field line it experiences a centrifugal force due to the field curvature, and therefore drifts perpendicular to both the centrifugal force and $\mathbf{B}$ as described in Section 1.4. Defining the radius of curvature $\mathbf{R}_c$ of the magnetic field lines as in Figure 1.4, then

$$\mathbf{F}_{curv} = \frac{m v_\parallel^2}{R_c^2} \mathbf{R}_c \tag{1.27}$$

and so

$$\mathbf{v}_{curv} = \frac{m}{q} \frac{v_\parallel^2}{R_c^2} \frac{\mathbf{R}_c \times \mathbf{B}}{B^2} \ . \tag{1.28}$$

This curvature drift can be parallel or anti-parallel to convection electric fields, thereby leading to energy gains or losses, respectively, as for the ∇B drift discussed in Section 1.5.1. This is particularly relevant to shock-drift acceleration, discussed more below and in Lectures 5, 10, 11 and 13.

Gradients in a plasma's magnetic field are constrained by Ampere's Law $\nabla \times B = \mu_0 \mathbf{j}$ (neglecting the displacement current). This means that plasma particles are almost always subject simultaneously to both the ∇B and curvature drifts, not just one or the other. In particular, relating the radius of curvature to the magnetic field using Ampere's Law and **assuming no plasma currents** (invalid for a shock wave but valid for most of Earth's magnetosphere), Eq. (1.28) may be rewritten

$$\mathbf{v}_{curv} = \frac{m}{q} \frac{v_\parallel^2}{B} \frac{\mathbf{B} \times \nabla B}{B^2} \ . \tag{1.29}$$

Combining Eqs (1.25) and (1.29), the combined drift is

$$\mathbf{v}_B = \frac{m}{2qB} (v_\perp^2 + 2v_\parallel^2) \frac{\mathbf{B} \times \nabla B}{B^2} \ . \tag{1.30}$$

This drift velocity naturally leads to charge separations and currents in plasmas, as well as dispersion of particles with different parallel & perpendicular energies, charges, and masses.

1.5.3 Applications

The ∇B and curvature drifts mean that it is not possible to confine a plasma using curved magnetic fields or, more generally, in magnetic field configurations such that $\mathbf{B} \times \nabla B \neq 0$. One reason is that the charge-dependent drift $\mathbf{v}_B$ causes charge separations and the build up of an ambipolar electric field perpendicular to $\mathbf{B}$, which then leads to an $E \times B$ drift of the plasma across the magnetic field. These problems are of great interest in laboratory and fusion plasma physics.

These drifts are very important in understanding the motions of particles in the solar wind and Earth's magnetosphere. For instance, the ∇B and curvature drifts are important in understanding particle acceleration at shock waves and current sheets (where it is reiterated that $\mathbf{j} \neq 0$, so that (1.30) is invalid), as well as in the injection of energetic particles close to Earth during magnetic substorms.

One specific illustration of how these drifts lead to particle acceleration involves shock waves, which have increases in magnetic field strength and direction across the shock (Figure 1.5). Consider the solar wind flow onto Earth's bow shock, in particular. The drift $\mathbf{v}_{\nabla B}$ is into the page for protons and out of the page for electrons. Notice now that the solar wind's convection electric field is into the page. Accordingly, the proton drift velocity $\mathbf{v}_{\nabla B}$ is parallel to $\mathbf{E}_{sw}$ while for electrons $\mathbf{V}_{\nabla B,e}$ is anti-parallel to $\mathbf{E}_{sw}$. In both cases, the drifting particles can gain energy, consistent with Eq. (1.4). This mechanism is called **shock-drift acceleration**. It is important in understanding energetic particles in the solar corona, interplanetary medium, and probably the outer heliosphere, as well as in Astrophysics. (The figure also shows that curvature drifts lead to energy losses in this case.)

1.6 Adiabatic invariants

For periodic motions the theory of mechanics shows that quantities called actions can remain invariant for slow changes in the system. An action J can be defined in terms of generalized coordinates q_{gen} and conjugate momenta p_{gen} by

$$J = \int p_{gen} dq_{gen} \ , \tag{1.31}$$

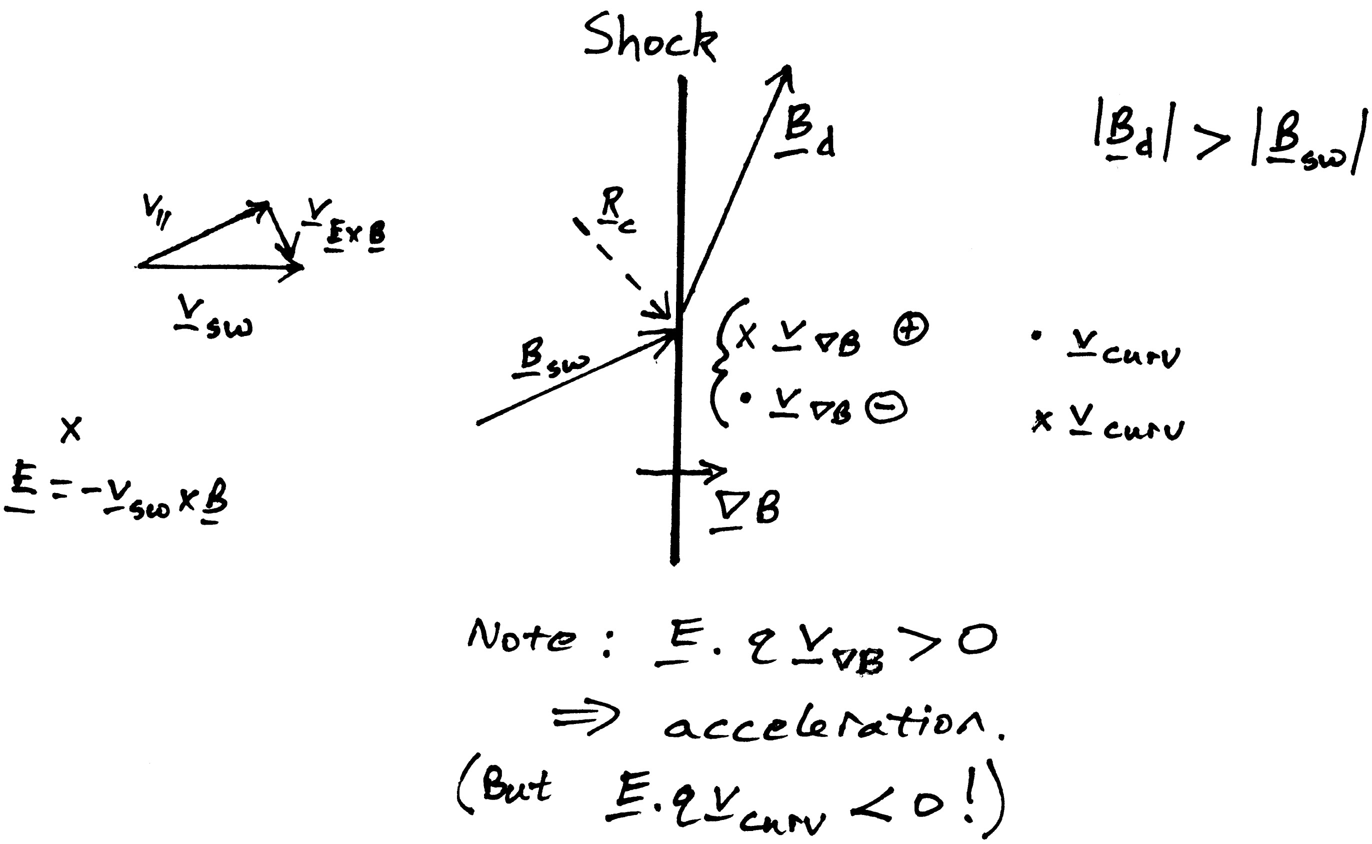

Figure 1.5: Illustration of the $\mathbf{E} \times \mathbf{B}$, gradient and curvature drifts relevant to a fast mode shock wave like Earth's bow shock. Note that the gradient drift leads to acceleration of the particles by the solar wind's convection electric field $\mathbf{E}_{sw} = -\mathbf{v}_{sw} \times \mathbf{B}_{sw}$, but that the curvature drift leads to energy losses.

where the integral is over one period of the motion. A particle's gyromotion is one example of a periodic motion amenable to the construction of an approximate constant of the motion (or invariant). Defining q_{gen} to be the gyrophase ϕ, then the angular momentum $p_{gen} = l = mv_\perp r_L$ is the conjugate momentum. Inserting these variables into (1.31) and integrating one finds

$$J = 2\pi \frac{m}{q}\frac{v_\perp^2}{B} = \frac{4\pi m}{q}\mu \tag{1.32}$$

for slowly varying B. The quantity

$$\mu = \frac{v_\perp^2}{2B} \tag{1.33}$$

is known as the **first adiabatic invariant** of a plasma particle. This implies that a particle's perpendicular energy $W_\perp = 1/2mv_\perp^2$ is proportional to B if μ is constant.

Another, perhaps more obvious derivation of the first adiabatic invariant is as follows. Assume that the particle sees a small change in $\mathbf{B}$ during a gyroperiod, whether due to temporal or spatial variations in $\mathbf{B}$. I.E.,

$$\frac{1}{\Omega_c}|\frac{\partial B}{\partial t}B^{-1}| \ll 1 \ . \tag{1.34}$$

The change in $W_\perp$ in one gyroperiod is

$$\Delta W_\perp = q\int \mathbf{E}.\mathbf{d}l \tag{1.35}$$

$$= \frac{\partial}{\partial t}\left(q\int_S \mathbf{B}.d\mathbf{S}\right) \ . \tag{1.36}$$

Assuming the orbit size changes very little in one gyroperiod then

$$\Delta W_\perp \approx q\pi r_L^2 \frac{\partial B}{\partial t}. \tag{1.37}$$

Since the change in B in one gyroperiod is

$$\Delta B = \frac{2\pi}{\Omega_c}\frac{\partial B}{\partial t} \tag{1.38}$$

then $\Delta W_\perp = W_\perp \Delta B/B$ or

$$\Delta\left(\frac{W}{B}\right) = 0 = \Delta(\mu) \ . \tag{1.39}$$

That is, μ is a constant.

Other adiabatic invariants also exist. The **second** or **longitudinal** adiabatic invariant is associated with the periodic bouncing of particles in magnetic flux tubes and magnetic bottles. Here

$$J_L = m\int v_\parallel dz \tag{1.40}$$

A third adiabatic invariant can be associated with the periodic drift of a particle (due to ∇B and curvature drifts) around a dipole magnetic field. It is useful for studying particle motions in Earth's magnetosphere but is not addressed further here.

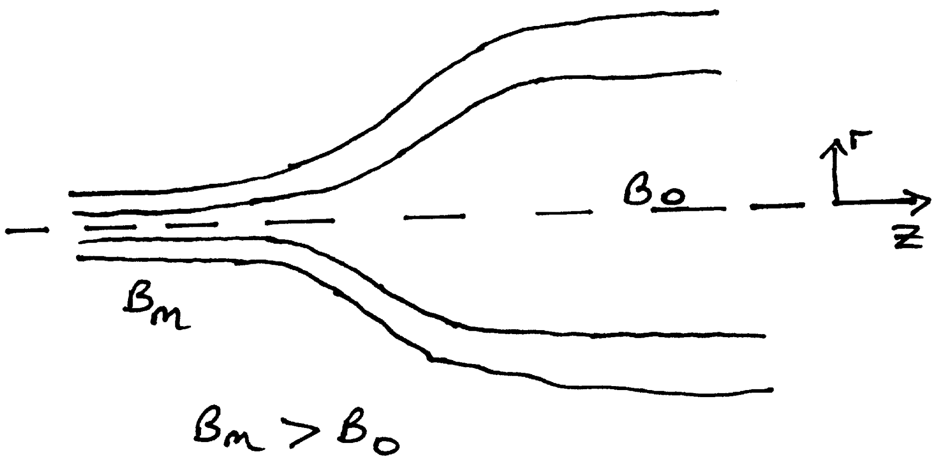

Figure 1.6: A magnetic bottle, with maximum field B_m at large negative z and minimum field B_0 in the center. Particles move with constant μ and energy in this bottle, so that particles with pitch-angles in the loss cone are lost from the bottle.

1.7 Magnetic Mirrors: the effects of $\nabla B \parallel \mathbf{B}$

This section addresses the mirroring properties of a longitudinal gradient in the magnetic field; i.e., the effects of gradients in magnetic field strength parallel to **B**. Two ways of doing this are: first, using conservation of μ and the total kinetic energy, and second, by considering the forces acting on the particle.

Figure 1.6 shows a cylindrically symmetric situation with $\nabla B \parallel \mathbf{B}$ (half of a so-called magnetic bottle). Note the field lines becoming closer together as $|\mathbf{B}|$ increases. For slow gradients and time-independent fields $\mu = v_\perp^2/2B$ is constant and there is no electric field **E**, so that the kinetic energy $m(v_\parallel^2 + v_\perp^2)/2$ is a constant. Thus

$$v_\parallel^2(z) = v_\parallel^2(0) - v_\perp^2(0)\left[\frac{B(z)}{B(0)} - 1\right] \tag{1.41}$$

since

$$\frac{v_\perp^2(z)}{B(z)} = \frac{v_\perp(0)^2}{B(0)} \ . \tag{1.42}$$

Thus the particle's parallel speed decreases as it moves into the region with increased B, and may actually vanish at some point (**the magnetic mirror point**). This point depends on the initial parallel and perpendicular speeds and the fractional increase in B. A particle reaching its magnetic mirror point is reflected and retraces its trajectory - note that Eq. (1.41) is for $v_\parallel(z)^2$.

The pitch angle α of a particle is defined by

$$\tan\alpha = \frac{v_\perp}{v_\parallel} \tag{1.43}$$

with

$$\sin\alpha = v_\perp/v \ . \tag{1.44}$$

Exercise 1.4: Use conservation of the magnetic moment and energy to show that a particle will be reflected from a (slow) magnetic field gradient if

$$\sin^2\alpha \geq \frac{B(0)}{B(z)} \ . \tag{1.45}$$

The other approach toward magnetic mirroring is to directly study the forces on a particle. In the case of axisymmetric fields, $\mathbf{B}_\theta = 0$ and $\mathbf{B}_z$ and $\mathbf{B}_r$ are related by $\nabla.\mathbf{B} = 0$. When $\partial B_z/\partial z$ is slowly varying one may integrate the equation $\nabla.\mathbf{B} = 0$ to obtain

$$B_r = -\frac{r}{2}\frac{\partial B}{\partial z} \ . \tag{1.46}$$

Then

$$F_{\|} = qv_{\perp}B_r = -\mu\frac{\partial B}{\partial z} \ . \tag{1.47}$$

This equation shows that the particle experiences a decrease in $v_{\|}$, is potentially reflected, and (for constant μ and kinetic energy) increases in $v_{\perp}$ when it enters a region with larger B.

Note that not all particles entering a region with increased B will be reflected. Instead, those with $sin^2\alpha < B(0)/B(z)$ will not be reflected but will instead pass through the magnetic enhancement. Define B_m, the maximum field in the trap, to be the mirror field: then particles with

$$sin^2\alpha_0 = sin^2\left(\tan^{-1}[v_{\perp}(0)/v_{\|}(0)]\right) < \frac{B(0)}{B_m} \tag{1.48}$$

will not be mirrored. This leads to **loss cone** anisotropies in the particle distribution function. Importantly, these loss cone anisotropies can drive plasma waves and radio emissions which are observable, including Earth's Auroral Kilometric Radiation and various solar emissions. Moreover, generation of the waves and scattering by the wave fields drive the plasma particles into the loss cone, leading to loss of plasma. Magnetic mirroring can also lead to particle energisation, especially at shock waves.