Quantum computing with a reasonable overhead

Simplified quantum compiling with complex gate distillation

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Outline

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How to rotate a qubit by 0.23π?

- Along the z axis: just wait for $t = 0.23\pi\hbar/J$ seconds.
- Along the x axis: just Rabi pulse the qubit for $t = 0.23\pi\hbar/A$ seconds.
- General by Euler angle decomposition.

How about errors?

- Algorithm uses 10^6 qubits and has dept 10^8.
- There are 10^{14} occasions to pick up errors

$$|\psi_t\rangle = |\phi_t^0\rangle + |E_t\rangle = (U_t + \epsilon V_t)(|\phi_{t-1}^0\rangle + |E_{t-1}\rangle)$$

$$\Rightarrow |E_t\rangle = \epsilon V_t|\phi_{t-1}^0\rangle + (U_t + \epsilon V_t)|E_{t-1}\rangle$$

$$\Rightarrow |E_t| \leq \epsilon + |E_{t-1}|$$

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- Each gate requires accuracy $\ll 10^{-14}$.

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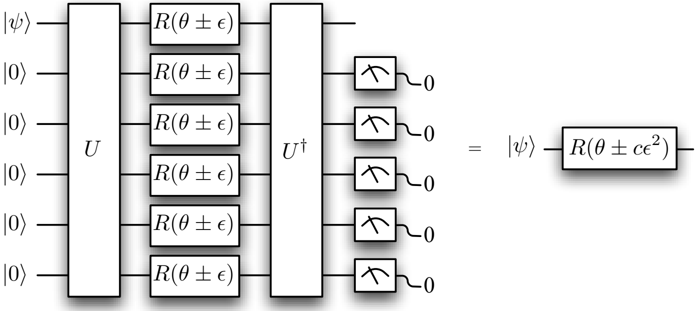
Transversal gates



- This only works for very special angles θ.
- Error can be further suppressed by iterating (concatenation).
- We can realize the Clifford group this way: CNOT, H, P = $Z^{1/2}$.
- U itself is Clifford, so iterations don't introduce more errors.
- Not a universal gate set.
- Slightly more general setting admits T= $Z^{1/4}$, universal.

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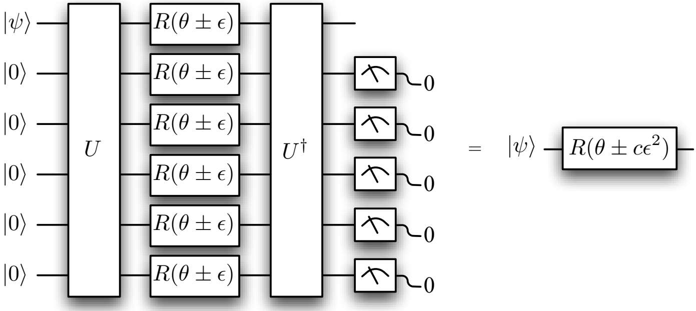

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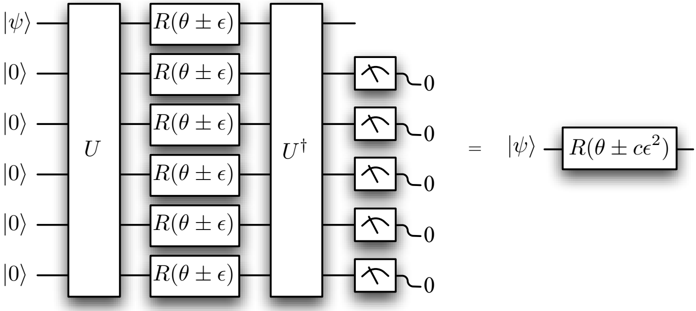

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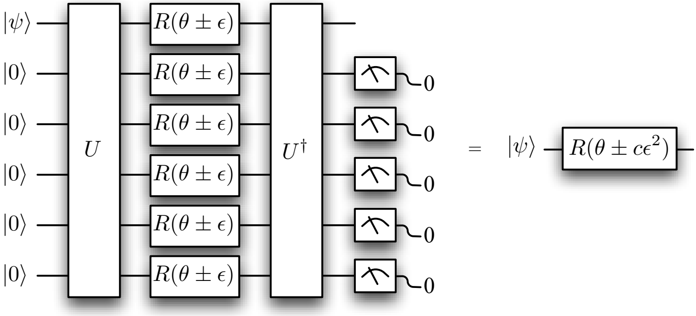

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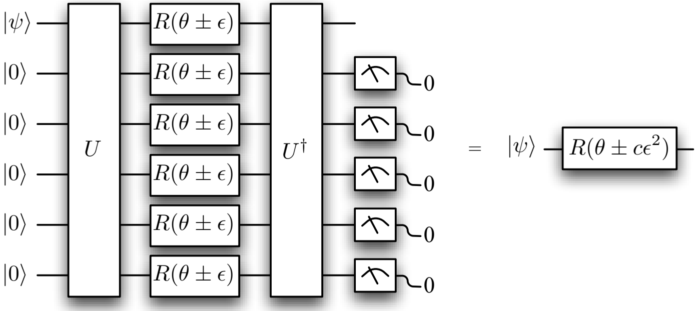

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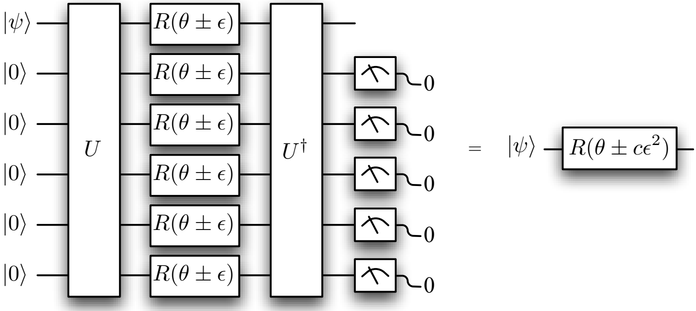

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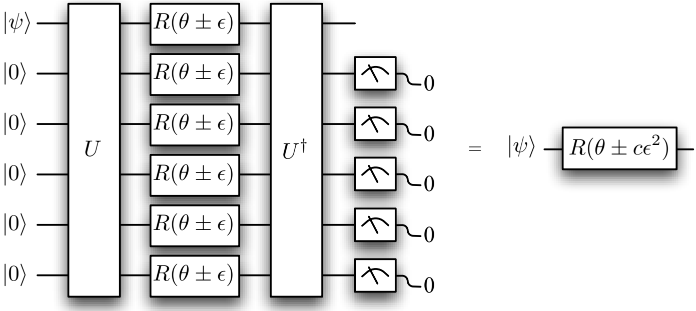

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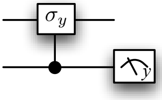
State injection

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|Y_\theta\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$$



$$\cos\frac{\theta}{2}|0\rangle|\psi\rangle + \sin\frac{\theta}{2}|1\rangle\sigma_y|\psi\rangle$$

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Can realize a rotation around y of angle $\pm\theta$ given state $|Y_\theta\rangle$ and Clifford operations.

- An error $\theta \pm \epsilon$ in the state $|Y_\theta\rangle$ translate into an error ϵ in the gate.

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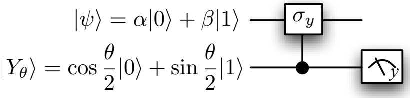

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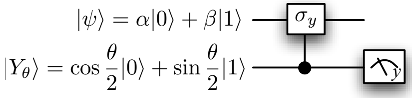

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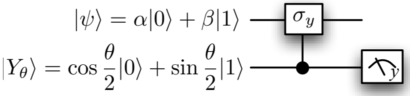

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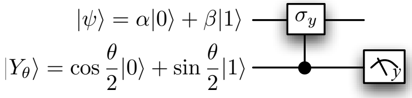

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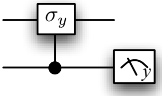

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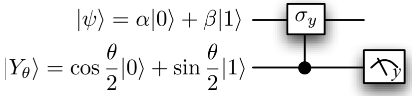

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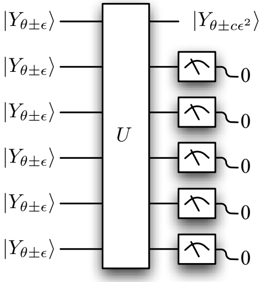

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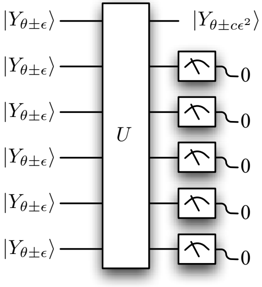

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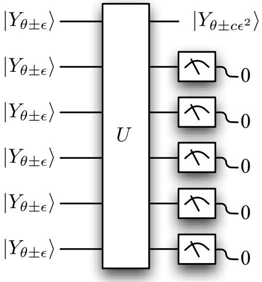

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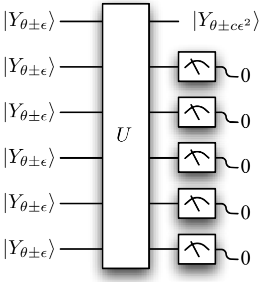

- U is Clifford.
- This only works for very special angles θ.
- With Clifford operations, all we need is $\theta = \pi/8$ to get T = $Z^{1/4}$, universal.

Cost for physical noise rate 1%

Precision	# inputs
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10^{-6}	28
10^{-8}	87
10^{-10}	139
10^{-12}	261
10^{-15}	436
10^{-18}	697
10^{-23}	1309
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10^{-38}	3632
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Meier, Eastin, & Knill

- This is the number of noisy $|Y_{\pi/8}\rangle$ states needed to distill one high-accuracy $|Y_{\pi/8}\rangle$.
- It is assumed that Clifford operations are noiseless.

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Outline

Compilation

How to rotate a qubit by 0.23π?

- Get a universal set of gates CNOT, H, P = Z$^{1/2}$, T = Z$^{1/4}$.
  - Because they are "transversal" in the code.
  - By distilling $|Y_{\pi/8}\rangle$.
- Compile: R(0.23π) $\approx$HTHPTPTHTHPTPTPTHPTHPHTPHTPTH
- Precision δ requires $\mathcal{O}(\log^c \frac{1}{\delta})$ gates (Solovay-Kitaev).
- Hidden constant in $\mathcal{O}$ are huge.

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Kliuchnikov, Maslov, & Mosca

- Need precision 10^{-14}
- Compiled logical gates will use 10^4 T gates
- Each of these T gates must have accuracy $\frac{10^{-14}}{10^4} = 10^{-18}$

Overhead = $14,312 \times 697 = 9,975,464$

Clifford operation cost not accounted.

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Outline

High level state distillation

Our approach

- High level magic-state distillation $|Y_k\rangle = |Y_\theta\rangle$ with $\theta = 2\pi/2^k$.
- Corresponding rotations $R_k = R(2\pi/2^k)$.
    - T = R_3.
- Note that $R_{k-1}Z|Y_k\rangle = |Y_k\rangle$.
- Need R_{k-1} to distil gates $|Y_k\rangle$ (Gottesman-Chuang).
- Compiling becomes trivial (next slide).

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- We use the 4-qubit code in a similar way as Meier, Eastin, & Knill.

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How to rotate a qubit by 0.23π?

$$0.23\pi = 2\pi \times 0.000111010111000010100011110101111$$

- Rotate to precision 2^k with $k+1$ of these rotations.
- Get any single-qubit rotation by Euler angles decomposition.

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$$R(2\pi/2^{32})$$

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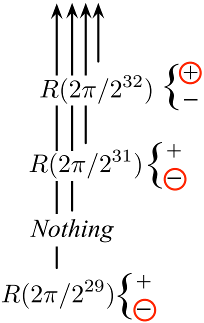

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Clifford (deterministic)

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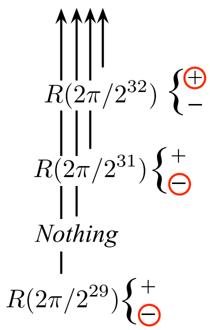

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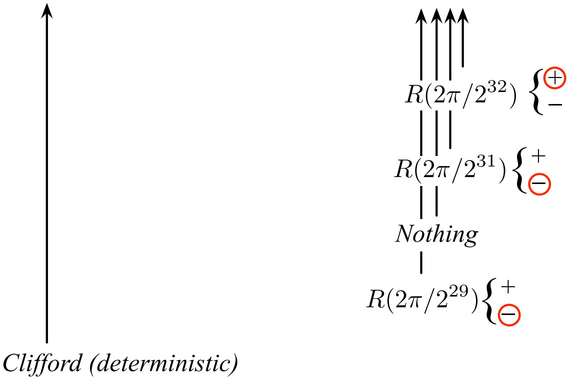

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- Get any single-qubit rotation by Euler angles decomposition.

Twirl

- For a qubit basis $\{|\psi\rangle, |\overline{\psi}\rangle\}$, twirl $T|\psi\rangle = |\psi\rangle$ and $T|\overline{\psi}\rangle = -|\overline{\psi}\rangle$
- Twirl makes matrices diagonal in $\{|\psi\rangle, |\overline{\psi}\rangle\}$ basis

$$\frac{1}{2}(\rho + T\rho T^{\dagger}) = \langle\psi|\rho|\psi\rangle \cdot |\psi\rangle\langle\psi| + \langle\overline{\psi}|\rho|\overline{\psi}\rangle \cdot |\overline{\psi}\rangle\langle\overline{\psi}|$$

- Twirl for $\{|0\rangle, |1\rangle\}$ is $T_0 = Z$.
- Twirl for $\{|Y_k\rangle, |\overline{Y}_k\rangle\}$ is $T_k = R_{k-1}Z$
- We can perform T_k with a $|Y_{k-1}\rangle$ state.

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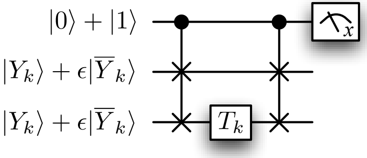
- We can perform T_k with a $|Y_{k-1}\rangle$ state.

Swap test



$$(|0\rangle + |1\rangle)|Y_k\rangle|Y_k\rangle \xrightarrow{SWAP} |0\rangle|Y_k\rangle|Y_k\rangle + |1\rangle|Y_k\rangle|Y_k\rangle$$

$$\xrightarrow{T_k} |0\rangle|Y_k\rangle|Y_k\rangle + |1\rangle|Y_k\rangle|Y_k\rangle$$

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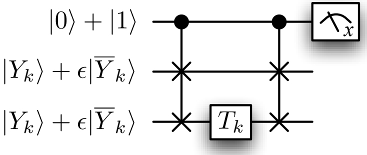
$$\xrightarrow{T_k} |0\rangle|\overline{Y}_k\rangle|Y_k\rangle - |1\rangle|Y_k\rangle|\overline{Y}_k\rangle$$

$$\xrightarrow{SWAP} |0\rangle|\overline{Y}_k\rangle|Y_k\rangle - |1\rangle|\overline{Y}_k\rangle|Y_k\rangle$$

$$= (|0\rangle - |1\rangle)|\overline{Y}_k\rangle|Y_k\rangle$$

Swap test



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\begin{aligned}
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&= (|0\rangle + |1\rangle)|Y_k\rangle|Y_k\rangle
\end{aligned}
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(|0\rangle + |1\rangle)|\overline{Y}_k\rangle|Y_k\rangle &\xrightarrow{SWAP} |0\rangle|\overline{Y}_k\rangle|Y_k\rangle + |1\rangle|Y_k\rangle|\overline{Y}_k\rangle \\
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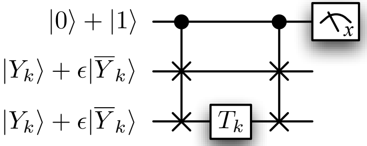

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(|0\rangle + |1\rangle)|\overline{Y}_k\rangle|Y_k\rangle &\xrightarrow{SWAP} |0\rangle|\overline{Y}_k\rangle|Y_k\rangle + |1\rangle|Y_k\rangle|\overline{Y}_k\rangle \\
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&= (|0\rangle - |1\rangle)|\overline{Y}_k\rangle|Y_k\rangle
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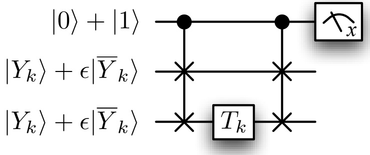
Swap test



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(|0\rangle + |1\rangle)|\overline{Y}_k\rangle|Y_k\rangle &\xrightarrow{SWAP} |0\rangle|\overline{Y}_k\rangle|Y_k\rangle + |1\rangle|Y_k\rangle|\overline{Y}_k\rangle \\
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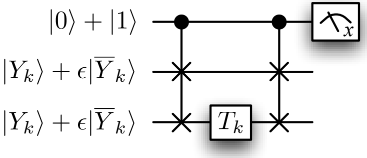

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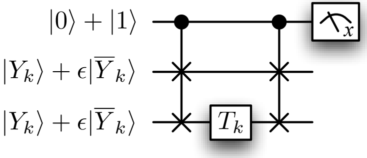

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&\xrightarrow{T_k} |0\rangle|\overline{Y}_k\rangle|Y_k\rangle - |1\rangle|Y_k\rangle|\overline{Y}_k\rangle \\
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&= (|0\rangle - |1\rangle)|\overline{Y}_k\rangle|Y_k\rangle
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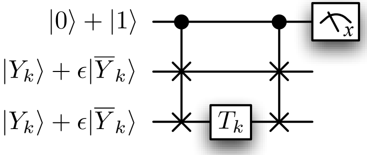
Swap test



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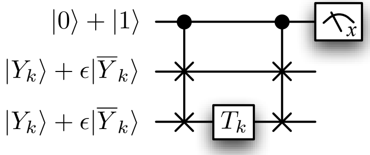

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Swap test

- Get outcome $+$ with probability $1 - 2\epsilon$.
- Given this outcome, state is $|Y_k\rangle|Y_k\rangle + \epsilon^2|\overline{Y}_k\rangle|\overline{Y}_k\rangle$.
- Get outcome $-$ with probability 2ϵ, reject the state.

Distillation

Can quadratically increase the fidelity of $|Y_k\rangle$ using T_k and controlled SWAP gate.

- Works as well for incoherent noise $|Y_k\rangle\langle Y_k| + \epsilon|\overline{Y}_k\rangle\langle\overline{Y}_k|$.
  - This can be obtained from any state by twirling.
- How to perform T_k?
  - Answer: By injecting distilled $|Y_{k-1}\rangle$ states.
- How to perform controlled-SWAP without introducing more errors?
  - Not a Clifford operation.

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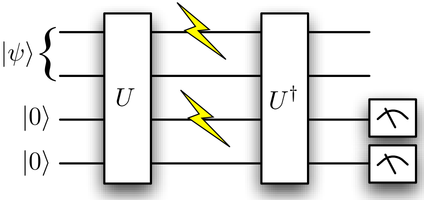
4 qubit code

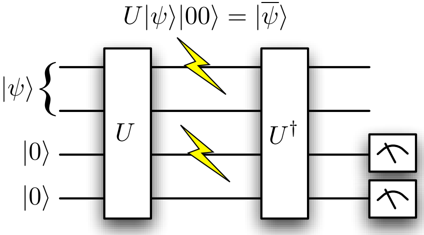
$$U|\psi\rangle|00\rangle = |\overline{\psi}\rangle$$



$$\overline{Z}_1 \equiv UZ_1U^\dagger = ZIIZ$$
$$\overline{X}_1 \equiv UX_1U^\dagger = XXII$$
$$\overline{Z}_2 \equiv UZ_2U^\dagger = XIIX$$
$$\overline{X}_2 \equiv UX_2U^\dagger = ZZII$$
$$S_1 \equiv UZ_3U^\dagger = ZZZZ$$
$$S_2 \equiv UZ_4U^\dagger = XXXX$$

- U is Clifford.
- $Z|0\rangle = |0\rangle \Leftrightarrow S|\overline{\psi}\rangle = |\overline{\psi}\rangle$.
- This code detects all single-qubit errors.
  - Error rate $\epsilon \to \epsilon^2$.
- Apply H to all qubits maps S to itself.
  - $H^{\otimes 4}\overline{Z}_1H^{\otimes 4} = XIIX = \overline{Z}_2$
  - $H^{\otimes 4}\overline{X}_1H^{\otimes 4} = ZZII = \overline{X}_2$
  - $H^{\otimes 4}\overline{Z}_2H^{\otimes 4} = ZIIZ = \overline{Z}_1$
  - $H^{\otimes 4}\overline{X}_2H^{\otimes 4} = XXII = \overline{X}_2$
- Transversal Hadamard realizes SWAP.

4 qubit code



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$$S_2 \equiv UZ_4U^\dagger = XXXX$$

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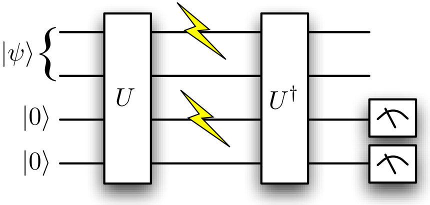

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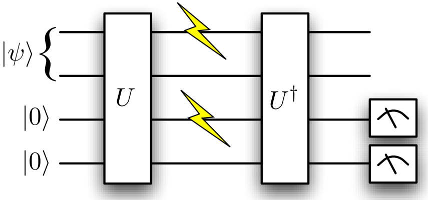

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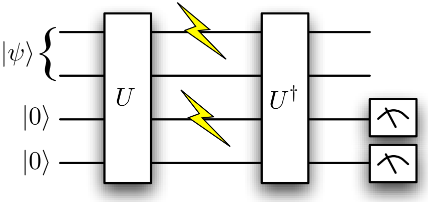

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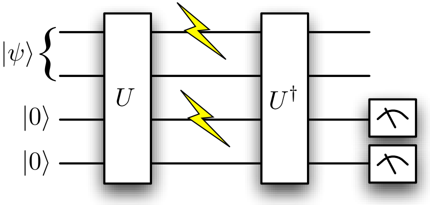

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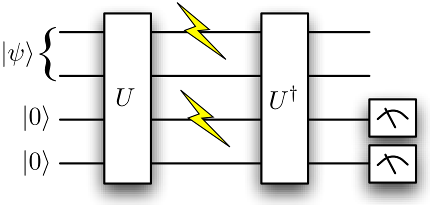

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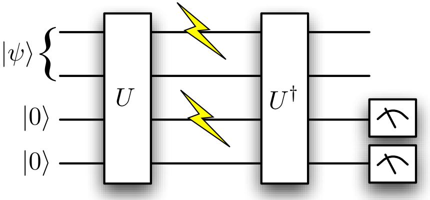

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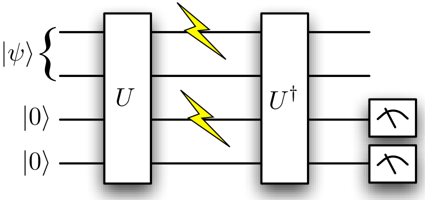

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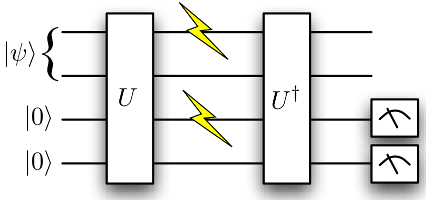

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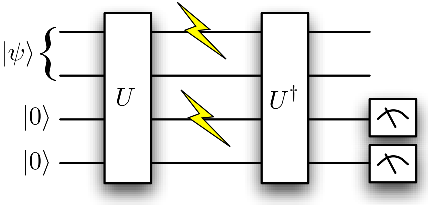

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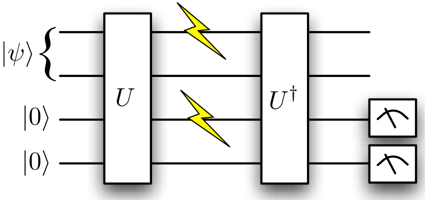

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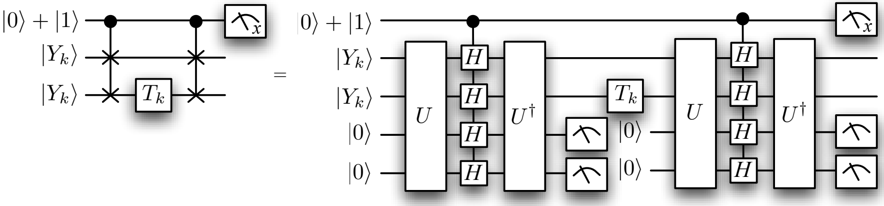

- How to realize controlled-H gates? (Not Clifford)
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- But $|Y_3\rangle$ gates are noisy!
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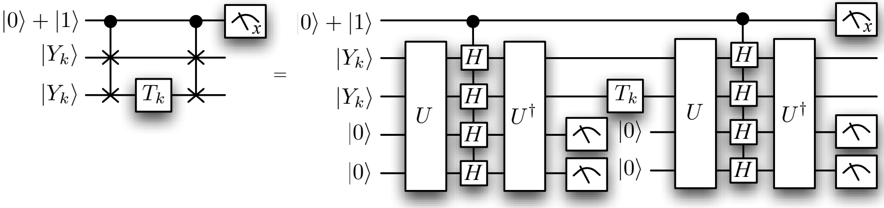

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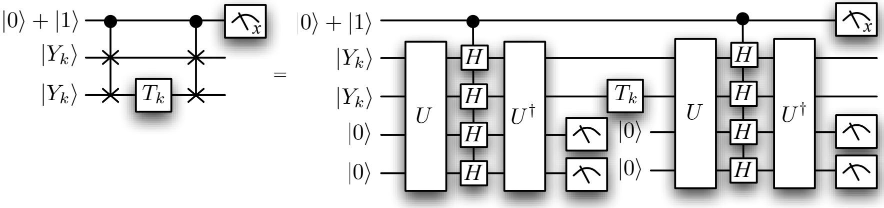

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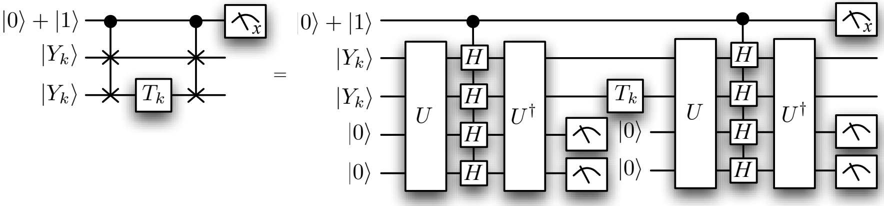

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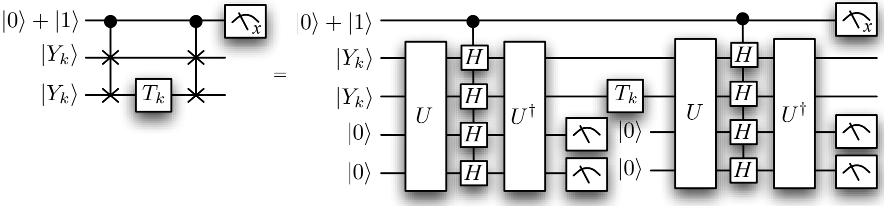

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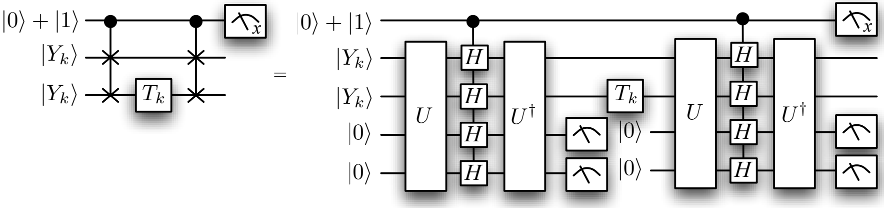

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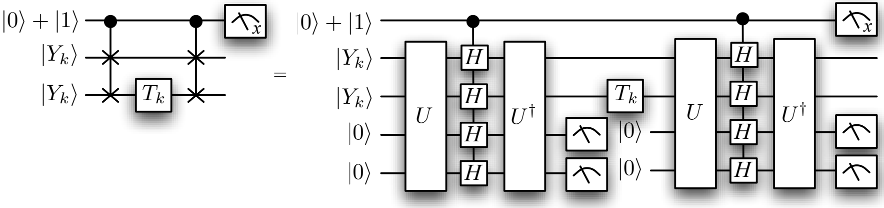

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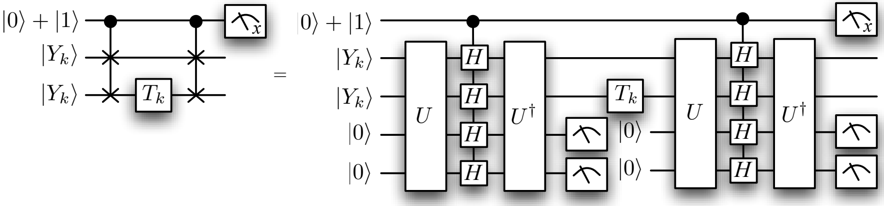

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To distill a $|Y_k\rangle$ state, we need...

- Some noisy versions of $|Y_k\rangle$.
- Some “not so noisy" versions of $|Y_{k-1}\rangle$ to implement the twirl T_k.
  - As used in the SWAP test.
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Given the accuracy $1-\epsilon_j$ of the inputs $|Y_j\rangle$, we can...

- Compute the accuracy of the distilled $|Y_k\rangle$.
- Compute the rejection probability (wasting these inputs)
  - Reject when the 4-qubit code detects an error.
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- Knowing the rejection probability enables us to compute the average number of each component used.
- In our calculations, the accuracy of $|Y_j\rangle$ has two components (diagonal and off-diagonal).

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Outline

Rule of thumb

- Distillation with perfect gates takes ϵ_k to $\epsilon'_k = c\epsilon_k^2$.
- The use of noisy $|Y_j\rangle$ to distill $|Y_k\rangle$ will deteriorate this ϵ'_k.
- How accurate should the $|Y_j\rangle$ be?
  - Having $\epsilon_j = 10^{-30}$ while attempting to distill $\epsilon'_k = 10^{-10}$ seems like an overkill, we've wasted our time obtaining very high-quality $|Y_j\rangle$.
  - If they are too noisy, distillation will be useless.
- If N_j states $|Y_j\rangle$ are used in the distillation, use $\epsilon_j \approx \epsilon'_k/N_j$.
  - This will roughly double ϵ'_k.
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Initial accuracy

- Standard assumption: $|Y_3\rangle$ is initially prepared with accuracy 1%, and distilled to any desired accuracy.
- How well should we assume $|Y_j\rangle$ can be prepared?
- Does it make sense to prepare $|Y_{10}\rangle \approx |0\rangle + 2^{-10}|1\rangle$ to accuracy 1%?
- May as well prepare $|0\rangle$.

Assumption on initial accuracy

- We use the scheme of Meier, Eastin, & Knill to prepare $|Y_3\rangle$ assuming an initial error of 1%.
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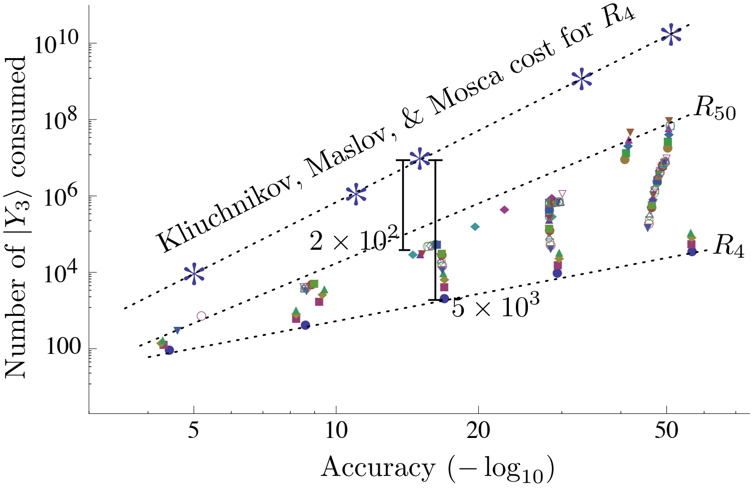
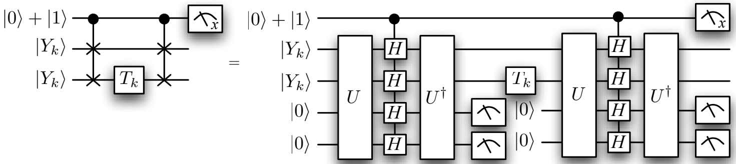
Results: cost of realizing R_k



Outline

High rate generalization



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- Gates c-H are realized by injecting states $|Y_3\rangle$: rate 1/2.
- Higher rate encoding?
- Use $[[n = 2m+2, k = 2m, d = 2]]$ with the property that $H^{\otimes n}$ SWAPs qubit j with $m + j$.

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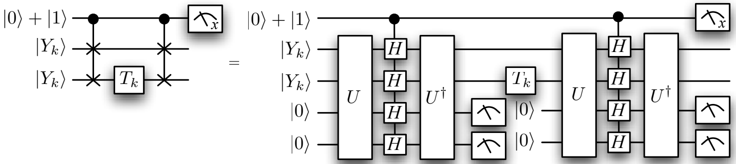

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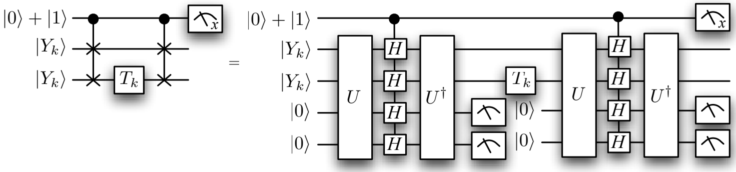

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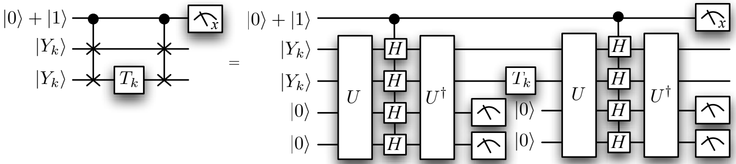

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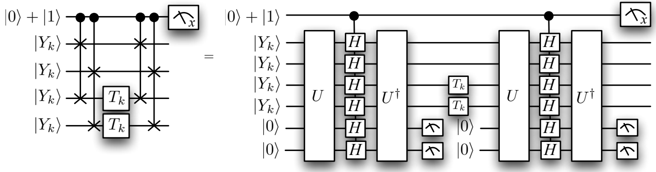

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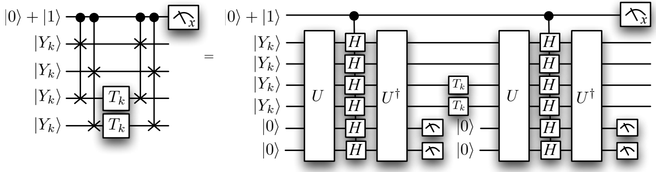

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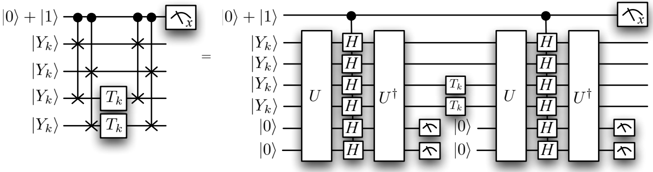

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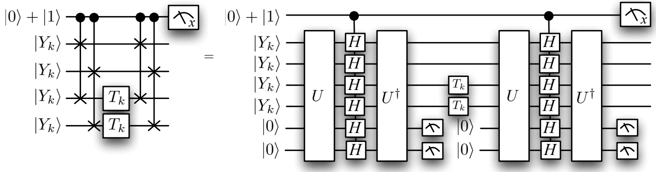

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- Gates c-H are realized by injecting states $|Y_3\rangle$: rate 1/2.
- Higher rate encoding?
- Use $[[n = 2m+2, k = 2m, d = 2]]$ with the property that $H^{\otimes n}$ SWAPs qubit j with $m + j$.

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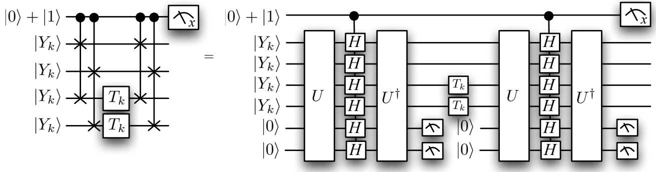

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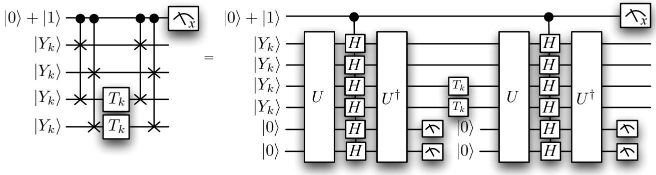

- Rate $= \frac{m}{m+1} \to 1$.
- Error suppression $\epsilon \to c\epsilon^2$
  - $c \propto m^2$ since there are more locations for failures.
- Rejection probability $p \propto m\epsilon$
  - Increase m along the distillation flow.
- Gain is more that just factor of 2 because of the recursive nature of the distillation.
- Enables to smoothly vary target accuracy.

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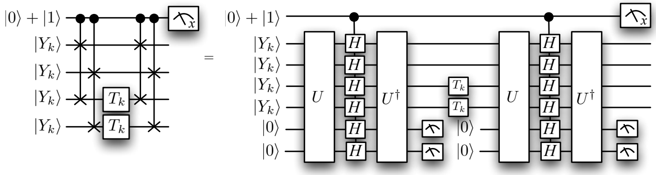

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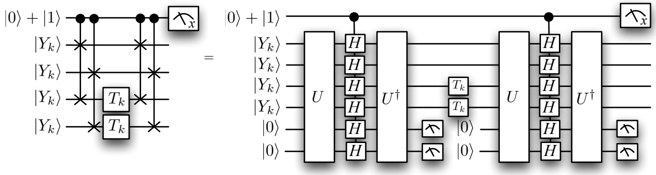

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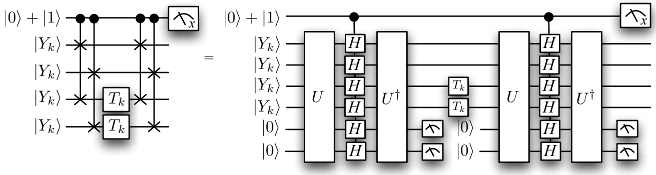

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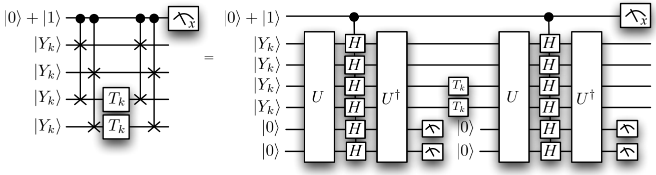

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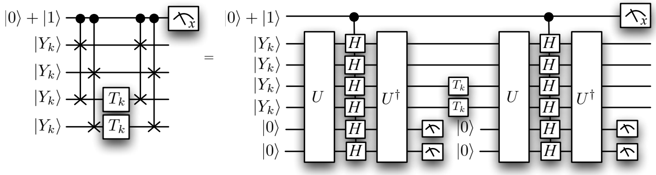

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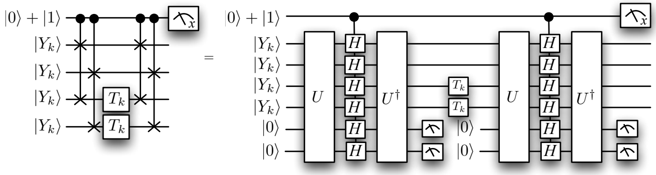

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  - Found savings of a 2-4 orders of magnitude for relevant noise regimes.
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