Supersymmetric Dark Matter Candidate (Neutralino) from a Fundamental Statistical Theory

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The fact that a Euclidean path integral is equivalent to a partition function suggests that the most fundamental description of Nature may be statistical.

Here we outline how a fundamental statistical theory can lead to standard physics, including supersymmetry and a supersymmetric dark matter candidate (as usual, a neutralino).
1 An initial comment on Euclidean versus Lorentzian path integrals

A Euclidean path integral

$$Z_E = \int \mathcal{D}(\text{Re} \phi_E) \mathcal{D}(\text{Im} \phi_E) e^{-S_E} , \quad S_E = \int d^D x \mathcal{L}_E$$

(1.1)

is formally transformed into a Lorentzian path integral

$$Z_L = \int \mathcal{D}(\text{Re} \phi_L) \mathcal{D}(\text{Im} \phi_L) e^{i S_L} , \quad S_L = \int d^D x \mathcal{L}_L$$

(1.2)

through an inverse Wick rotation

$$x_k^E = t_E \rightarrow ix_k^L = it_L.$$  

(1.3)

(Natural units with \( \hbar = c = 1 \) are used here, so \( S \) is dimensionless and \( t^0 = t \).)

\( S_L \) has the usual form of a classical action, and it leads to the usual description of quantized fields via path-integral quantization. In other words, the standard equations of physics follow from \( S_L \), and are therefore formulated in Lorentzian time. The Euclidean formulation, in either coordinate or momentum space, is ordinarily regarded as a mere mathematical tool which can simplify calculations and make them better defined.

Hawking, on the other hand, has suggested that Euclidean spacetime may actually be more fundamental than Lorentzian spacetime. On p. 74 of Hawking on the Big Bang and Black Holes (in a paper reprinted from Recent Developments in Gravitation), he states “In fact one could take the attitude that quantum theory and indeed the whole of physics is really defined in the Euclidean region and that it is simply a consequence of our perception of the universe that we interpret it in the Lorentzian regime.” And on p. 139 of his well-known popular book, A Brief History of Time, he again says “So maybe what we call imaginary time is really more basic, and what we call real is just an idea that we invent to help us describe what we think the universe is like.”
However, there is a fundamental problem with this point of view, because the factor $\delta^{S_L}$ in the Lorentzian formulation results in interference effects, whereas the factor $e^{-S_E}$ in the Euclidean formulation does not.

The work presented in the present talk has a different philosophy, and is also more concrete: We will start with a purely statistical picture at the most fundamental level, and will end with standard physics, including standard supersymmetry.

As usual, the most natural dark candidate is a neutralino. We are not able to offer any specific predictions for the parameters relevant to dark matter detection, but the justification for presenting the work at this conference is that it provides further confidence that standard supersymmetry is a robust description, which follows even from the quite unorthodox fundamental theory presented here.
2 Goals of the present theory: Explain the origins of

- Lorentz invariance
- bosonic fields
- fermionic fields
- supersymmetry
- gauge fields and their symmetry
- gravity
- quantum mechanics
- spacetime.
3 From a Statistical Picture to Standard Physics with Supersymmetry

(1) Statistical picture gives Euclidean-like "action" for bosons only (and no time yet):

$$Z = \int \mathcal{D}(\text{Re} \phi) \mathcal{D}(\text{Im} \phi) \ e^{-S_\phi} \ , \quad S_\phi = \int d^{D-1}x \mathcal{L}_\phi .$$ (3.1)

(2) Random fluctuations then give "action" with bosons, fermions, and a primitive supersymmetry:

$$Z = \int \mathcal{D}(\text{Re} \phi) \mathcal{D}(\text{Im} \phi) \mathcal{D}(\text{Re} \psi) \mathcal{D}(\text{Im} \psi) \ e^{-S} \ , \quad S = \int d^{D-1}x \mathcal{L} .$$ (3.2)

(3) Transformation of fields changes Euclidean-like factor $e^{-S}$ to Lorentzian-like factor $e^{iS}$:

$$Z = \int \mathcal{D}(\text{Re} \phi) \mathcal{D}(\text{Im} \phi) \mathcal{D}(\text{Re} \psi) \mathcal{D}(\text{Im} \psi) \ e^{iS} \ , \quad S = \int d^{D-1}x \mathcal{L} .$$ (3.3)

(4) Gravitational metric tensor and $SO(10)$ gauge fields, and their supersymmetric partners, result from rotations of vacuum state vector, in both 3-dimensional external space and $D = 4$ dimensional internal space.

(5) Time is defined by progression of 3-geometries in external space.

(6) Heisenberg equations of motion are then obtained for all fields.

(7) Transformation of bosonic fields gives standard supersymmetry.

(8) One finally obtains an effective action which is the same as that of standard physics with supersymmetry, except that particle masses are assumed to arise from supersymmetry breaking and radiative corrections.

In order to emphasize the more recent results first, we will discuss these steps in the following order: (7), (5), (3), (1), (2), (4), (6), (8).
4 Step (7): Transformation of bosonic fields gives standard supersymmetry.

In hep-th/0310039 and elsewhere, we presented a model for a fundamental theory which, starting with a simple statistical picture, leads to many interesting results, including a primitive form of supersymmetry. But the action for a fundamental bosonic field is found to have the form

$$S_b = \int d^4x \, \psi_b^\dagger \, i \sigma^\mu \partial_\mu \psi_b$$

at energies that are far below the Planck energy $m_P$ (with $\hbar = c = 1$) and in a locally inertial coordinate system. This is the conventional action for fermions, described by 2-component Weyl spinors, but it is highly unconventional for bosons, because a boson described by $\psi_b$ would have spin 1/2. The standard action for a Higgs field, which has proved so successful in the context of the Standard Model, has the quite different form appropriate for spin zero.

We now show that the original action above can be converted a form which is the same as that in standard physics with supersymmetry, corresponding to a spin zero boson $\phi$ and its auxiliary field $F$.

There are two distinct physical interpretations of this result, and here we do not specify which is correct: Either the physical fields (including Higgs fields) fundamentally have spin 1/2, and the spin-zero action obtained below is merely an effective action which yields the correct results for vector boson masses etc., or the conventional spin 0 plus auxiliary fields are fundamental. Which interpretation is correct essentially depends on the vacuum chosen by Nature. (I.e., the physical fields are those whose field operators give zero when operating on the vacuum state.) It is also potentially an issue to be resolved by experiment.

Even if the second interpretation is correct, however, there is still a major difference between the present theory and conventional physics: When time is defined in the way outlined below, it is defined in such a way that all initially spin 1/2 bosonic fields have positive energy (and the vacuum is thus stable). This ultimately means that these particles do not have antiparticles – an extremely unconventional result whose consistency needs to be checked (for example, in the context of the issues which originally motivated supersymmetry, like quadratic divergence of the Higgs mass and unification of coupling constants).

The predictions within the context of this conference, however, are perfectly conventional: All fermions, including neutralinos, have their standard properties.
We transform from the original 2-component field $\psi_b$ to two 1-component complex fields $\phi$ and $F$ by writing

$$\psi_b(x) = \psi^+ (x) + \psi^- (x) \quad (4.1)$$

$$\psi^+ (\vec{x}, t) = \sum_{\vec{p}, \omega} \phi (\vec{p}, \omega) \ u^+ (\vec{p}) \ e^{i\vec{p} \cdot \vec{x}} e^{-i\omega t} e^{-(\omega + |\vec{p}|)^{1/2}} \quad , \quad \psi^- (\vec{x}, t) = \sum_{\vec{p}, \omega} F (\vec{p}, \omega) \ u^- (\vec{p}) \ e^{i\vec{p} \cdot \vec{x}} e^{-i\omega t} e^{-(\omega + |\vec{p}|)^{-1/2}} \quad (4.2)$$

with

$$\vec{\sigma} \cdot \vec{p} \ u^+ (\vec{p}) = + |\vec{p}| \ u^+ (\vec{p}) \quad , \quad \vec{\sigma} \cdot \vec{p} \ u^- (\vec{p}) = - |\vec{p}| \ u^- (\vec{p}) \quad (4.3)$$

and

$$\phi (\vec{p}, \omega) = \int d^d x \ \phi (\vec{x}, t) e^{-i\vec{p} \cdot \vec{x}} e^{i\omega t} \quad , \quad F (\vec{p}, \omega) = \int d^d x \ F (\vec{x}, t) e^{-i\vec{p} \cdot \vec{x}} e^{i\omega t} . \quad (4.4)$$
Substitution then gives

$$S_0 = \sum_{\vec{p}, \omega} \left[ \phi^* \left( \vec{p}, \omega \right) \left( \omega^2 - |\vec{p}|^2 \right) \phi \left( \vec{p}, \omega \right) + F^* \left( \vec{p}, \omega \right) F \left( \vec{p}, \omega \right) \right]$$

$$= \int d^4 x \left[ \partial_{\mu} \phi^* \left( x \right) \partial_{\mu} \phi \left( x \right) + F^* \left( x \right) F \left( x \right) \right]$$

with $\partial_{\mu} = \eta_{\mu \nu} \partial_{\nu}$ and $\eta_{\mu \nu} = \text{diag} \left( -1, 1, 1, 1 \right)$. This, of course, has precisely the form of the action for a massless scalar boson field $\phi$ and its auxiliary field $F$.

With the fermionic action left in its original form, we have the standard form of the supersymmetric action for each pair of susy partners:

$$S_0 = \int d^4 x \left[ \psi^\dagger \left[ i \eta_{\mu \nu} \partial_{\nu} \psi_f + \partial_{\mu} \phi^* \left( x \right) \partial_{\mu} \phi \left( x \right) + F^* \left( x \right) F \left( x \right) \right] \right] .$$

Recall, however, that the expansion of the bosonic field includes only positive-frequency components.
5 Step (5): Time is defined by progression of 3-geometries in external space.

In the present picture, cosmological time is defined by the cosmic scale factor \( R \) (except that in principle there can be different branches for the state of the universe, corresponding to, e.g., expansion and contraction). More generally, the progression of time is locally defined by the progression of local 3-geometries.

An analogy: Consider a stationary state for a proton with coordinates \( \vec{X} \) passing a hydrogen atom with coordinates \( \vec{x} \). The time-independent Schrödinger equation can be written

\[
\left( -\frac{\hbar^2}{2m_p} \nabla^2_p + H_e \right) \Psi \left( \vec{X} \right) \psi \left( \vec{x}, \vec{X} \right) = E \Psi \left( \vec{X} \right) \psi \left( \vec{x}, \vec{X} \right)
\]

with \( \Psi \) required to satisfy

\[
-\frac{\hbar^2}{2m_p} \nabla^2_p \Psi \left( \vec{X} \right) = E \Psi \left( \vec{X} \right).
\]

Then the equation for \( \psi \) is

\[
\left( -\frac{\hbar^2}{2m_p} \nabla^2_p \Psi \cdot \nabla_p - \frac{\hbar^2}{2m_p} \nabla^2_p + H_e \right) \psi \left( \vec{x}, \vec{X} \right) = 0.
\]

The first term involves a local proton velocity

\[
\vec{v}_p = \hbar \nabla P \theta / m_p, \quad \Psi = |\Psi| e^{i\theta}.
\]

For a state in which the proton is moving rapidly, with

\[
\Psi = \Psi_0 e^{iP \cdot \vec{X}/\hbar},
\]

and in which \( (\hbar^2/2m_p) \nabla^2_p \psi \) is relatively small, we obtain

\[
\frac{i\hbar}{\partial t} \psi \left( \vec{x}, t \right) = H_e \psi \left( \vec{x}, t \right), \quad \frac{\partial}{\partial t} \equiv -\frac{\vec{P}}{m_p} \cdot \nabla_p.
\]
One then has an “internal time” defined within a stationary state.

Similarly, one can define time as a progression of 3-geometries.

This general idea of “time from no time” has been discussed by many authors for many years, but its role in the present theory is novel.

In the formulation of canonical quantum gravity obtained by DeWitt, following the classical formulation of Arnowitt, Deser, and Miser, the local canonical momentum operator

\[ \pi^i_j (\vec{x}) = -i \frac{\delta}{\delta g^{i j} (\vec{x})} \]

(5.7)

corresponds to the proton momentum operator \(-i\vec{P}\) in the analogy above. After introducing the 3-dimensional metric tensor in the way described below, we move from the original path-integral quantization to canonical quantization, with a state

\[ \Psi_{\text{total}} = \Psi_{\text{gravity}} [g^{i j} (\vec{x})] \Psi_{\text{otherfields}} [\phi_{\text{otherfields}}, g^{i j} (\vec{x})] \]

(5.8)

and time is defined essentially in the same way as in the analogy.

Consider a single complex scalar field $\phi_E$ with a 3-dimensional “Euclidean action” $S$:

$$Z_E = \int \mathcal{D}(\text{Re} \phi_E) \mathcal{D}(\text{Im} \phi_E) e^{-S}, \quad S = \int d^3 x \phi_E^* (\bar{x}) A \phi_E (\bar{x}).$$

(6.1)

In a discrete picture, the operator $A$ is replaced by a matrix with elements $A (\bar{x}, \bar{x}')$:

$$S = \sum_{\bar{x}, \bar{x}'} \phi_E^* (\bar{x}) A (\bar{x}, \bar{x}') \phi_E (\bar{x}').$$

(6.2)

$A$ can be diagonalized to $A (\bar{k}, \bar{k}') = a (\bar{k}) \delta_{\bar{k}, \bar{k}'}$. The “Euclidean path integral”

$$Z = \int \mathcal{D}(\text{Re} \phi_E) \mathcal{D}(\text{Im} \phi_E) e^{-S}$$

(6.3)

$$= \prod_{\bar{k}} \int_{-\infty}^{\infty} d (\text{Re} \phi_E (\bar{k})) \int_{-\infty}^{\infty} d (\text{Im} \phi_E (\bar{k})) \exp \left( - \sum_{\bar{x}, \bar{x}'} \phi_E^* (\bar{x}) A (\bar{x}, \bar{x}') \phi_E (\bar{x}') \right)$$

(6.4)

then becomes

$$Z = \prod_{\bar{k}} \int_{-\infty}^{\infty} d (\text{Re} \phi_E (\bar{k})) \int_{-\infty}^{\infty} d (\text{Im} \phi_E (\bar{k})) \exp \left( - \sum_{\bar{k}} \phi_E^* (\bar{k}) a (\bar{k}) \phi_E (\bar{k}) \right).$$

(6.5)
The Gaussian integrals over $\text{Re} \phi_E(\vec{k})$ and $\text{Im} \phi_E(\vec{k})$ may be evaluated as usual at each $\vec{k}$ to give

$$Z = \prod_{\vec{k}} \frac{\pi}{a(\vec{k})} = \frac{\prod_{\vec{k}} \pi}{\text{det} A} \quad (6.6)$$

Now let us define a path integral $Z'$ with new fields $\phi'_E$ and $\bar{\phi}'_E$, which are treated as independent and which vary along the real axis. In defining $Z'$, we include the formal Jacobian, with a value of $1/2$, for the transformation from $\text{Re} \phi_E$ and $\text{Im} \phi_E$ to $\phi'_E = \text{Re} \phi_E + i \text{Im} \phi_E$ and $\bar{\phi}'_E(x) = i(\text{Re} \phi_E - i \text{Im} \phi_E)$:

$$Z' = \left[ \prod_{\vec{k}} \frac{1}{2} \int_{-\infty}^{\infty} d(\phi'_E(\vec{k})) \int_{-\infty}^{\infty} d(\bar{\phi}'_E(\vec{k})) \right] \exp \left( \sum_{\vec{k}} i \bar{\phi}'_E(\vec{k}) a(\vec{k}) \phi'_E(\vec{k}) \right) \quad (6.7)$$

$$= \prod_{\vec{k}} \frac{1}{2} \int_{-\infty}^{\infty} d(\phi'_E(\vec{k})) \int_{-\infty}^{\infty} d(\bar{\phi}'_E(\vec{k})) \exp \left( i \bar{\phi}'_E(\vec{k}) a(\vec{k}) \phi'_E(\vec{k}) \right) \quad (6.8)$$

$$= \prod_{\vec{k}} \frac{1}{2a(\vec{k})} \int_{-\infty}^{\infty} d(\phi'_E(\vec{k})) \int_{-\infty}^{\infty} d(\bar{\phi}'_E(\vec{k})) 2\pi \delta \left( a(\vec{k}) \phi'_E(\vec{k}) \right) \quad (6.9)$$

$$= \prod_{\vec{k}} \frac{\pi}{a(\vec{k})} \quad (6.10)$$

$$= Z. \quad (6.11)$$
We could rewrite the expression defining $Z'$ in terms of contributions with $\phi_E^* \phi_E^*$ positive, and then regain the normal Lorentzian form by letting

$$
\phi^* \left( \vec{k} \right) \phi \left( \vec{k} \right) = \phi_E^* \left( \vec{k} \right) \phi_E \left( \vec{k} \right) \quad \text{or} \quad \left( \text{Re} \phi \left( \vec{k} \right) \right)^2 + \left( \text{Im} \phi \left( \vec{k} \right) \right)^2 = \phi_E^* \left( \vec{k} \right) \phi_E \left( \vec{k} \right) .
$$

(6.12)

However, here (and in the following) any two representations of the path integral are taken to be physically identical if they give the same result for all operators $A$ (including those which produce zero except for arbitrarily restricted regions of space and sets of fields). Then we can actually just note that “Gaussian” integrations can also be performed when $a \left( \vec{k} \right) \to -i a \left( \vec{k} \right)$, so that

$$
Z = \prod_k \frac{1}{\left( -i \right) \int_{-\infty}^{\infty} d\left( \text{Re} \phi \left( \vec{k} \right) \right) \int_{-\infty}^{\infty} d\left( \text{Im} \phi \left( \vec{k} \right) \right)} \exp \left( i \sum_k \phi^* \left( \vec{k} \right) a \left( \vec{k} \right) \phi \left( \vec{k} \right) \right)
$$

(6.13)

$$
= \prod_a \frac{1}{\left( -i \right) \int_{-\infty}^{\infty} d\left( \text{Re} \phi \left( \vec{x} \right) \right) \int_{-\infty}^{\infty} d\left( \text{Im} \phi \left( \vec{x} \right) \right)} \exp \left( i \sum_{\vec{x}, \vec{x}'} \phi^* \left( \vec{x} \right) A \left( \vec{x}, \vec{x}' \right) \phi \left( \vec{x}' \right) \right)
$$

(6.14)

or in the continuum limit

$$
Z = \int \mathcal{D} \left( \text{Re} \phi \right) \mathcal{D} \left( \text{Im} \phi \right) e^{iS} , \quad S = \int d\vec{x} \, \phi^* \left( \vec{x} \right) A \left( \vec{x} \right) \phi \left( \vec{x} \right) .
$$

(6.15)

where the factors of $\left( -i \right)$ have been absorbed in the definition of the Lorentzian path integral. Notice that all this is possible because time has not yet been introduced, so that the “Euclidean” and “Lorentzian” operators are just the same operator $A$. 

13
Now consider the addition of a simple interaction term:

$$S_E = \int d^4x \left[ \phi_E^* (\vec{x}) A \phi_E (\vec{x}) + b (\phi_E^* (\vec{x}) \phi_E (\vec{x}))^2 \right].$$

(6.16)

With the values of $\vec{x}$ again discrete, and $A$ again diagonalized, the original path integral becomes $Z = \prod_k z (\vec{k})$ with

$$z (\vec{k}) = \int_{-\infty}^{\infty} d(\text{Re} \phi_E (\vec{k})) \int_{-\infty}^{\infty} d(\text{Im} \phi_E (\vec{k})) e^{-i \phi (\vec{k})} \left[ (\text{Re} \phi (\vec{k}))^2 + (\text{Im} \phi (\vec{k}))^2 \right] e^{-b \left[ (\text{Re} \phi (\vec{k}))^2 + (\text{Im} \phi (\vec{k}))^2 \right]^2}$$

(6.17)

which has the form

$$z (\vec{k}) = \int d u e^{-au} e^{-bu^2}$$

(6.18)

where $u = \phi_E^* (\vec{k}) \phi_E (\vec{k})$ and $\int d u$ represents $\int_{-\infty}^{\infty} d(\text{Re} \phi_E (\vec{k})) \int_{-\infty}^{\infty} d(\text{Im} \phi_E (\vec{k}))$. Expanding the second factor above leads to an asymptotic series, but since this factor is only an approximation for small $u$ let us write

$$z (\vec{k}) = \int d u e^{-au} \sum_{n=0}^{\infty} \frac{1}{n!} (-bu^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n b^n}{n!} \frac{d^{2n}}{dau^{2n}} \int d u e^{-au}.$$

(6.19)

Let us next consider

$$z' (\vec{k}) = \int d u' e^{-au'} e^{-bu'^2}$$

(6.20)

where $u' = \phi'_E (\vec{k})$ and $\int du'$ represents $\int_{-\infty}^{\infty} d(\phi'_E (\vec{k})) \int_{-\infty}^{\infty} d(\phi'_E (\vec{k}))$. The above treatment can then be repeated to get

$$z' (\vec{k}) = \sum_{n=0}^{\infty} \frac{(-1)^n b^n}{n!} \frac{d^{2n}}{dau^{2n}} \int du' e^{-au'}.$$

(6.21)

But it was shown above that this last integral has the same value as $\int du e^{-au}$, so $z' (\vec{x}) = z (\vec{x})$. In other words, one can again transform from the original Euclidean fields $\text{Re} \phi_E$ and $\text{Im} \phi_E$ to new fields $\phi'_E$ and $\phi'_E$ which vary independently.
One can also repeat the other steps above to finally obtain

\[ Z = \int \mathcal{D}(\text{Re} \, \phi) \, \mathcal{D}(\text{Im} \, \phi) \, e^{iS} \]

(6.22)

\[ S = \int d^3 x \left[ \phi^* (\vec{x}) A \phi (\vec{x}) + b (\phi^* (\vec{x}) \phi (\vec{x}))^2 \right]. \]

(6.23)

The equivalence of the 3-dimensional “Euclidean” and “Lorentzian” pictures is thus not affected by interactions.

It is also possible to extend the treatment to more than one field, to fermion path integrals (over Grassmann variables), and to boson and fermion fields which are coupled to gauge fields and gravity.
Step (1): Statistical picture gives Euclidean-like “action” for bosons only (and no time yet)

\[ Z_E = \int \mathcal{D}(\text{Re } \phi) \mathcal{D}(\text{Im } \phi) \ e^{-S_{\phi}} \quad \text{and} \quad S_b = \int d^{D-1} x \mathcal{L}_b. \]  \hspace{1cm} (7.1)

We now outline how this \((D - 1)\)-dimensional precursor of the action follows from a simple microscopic and statistical picture. Our starting point is a single fundamental system which consists of \(N_w\) identical “whits”, with \(N_w\) variable. (“Whit”, whose meanings include “particle” and “least possible amount”, is an appropriate name for the irreducible objects that are postulated here.) Each whit can exist in any of \(M_w\) states, with the number of whits in the \(i\)th state represented by \(n_i\). A microstate of the fundamental system is specified by the number of whits and the state of each whit. A macrostate is specified by only the occupancies \(n_i\) of the states.

\(D\) of the states are used to define \(D\) coordinates \(x^M\) in Euclidean spacetime, \(m_w\) of the states are used to define observable fields \(\phi_k\), and the remaining \((M_w - m_w - D)\) states may be regarded as corresponding to fields that are unobservable (at least at the energy scales considered here).

An initial set of coordinates \(X^M\) is defined in terms of the occupancies \(n_M\):

\[ X^M = \pm n_M a_0 \]  \hspace{1cm} (7.2)

where \(M = 0, 1, \ldots, D - 1\). It is convenient to include a fundamental length \(a_0\) in this definition, so that the coordinates can later be expressed in conventional units. As one might expect, \(a_0\) eventually turns out to be comparable to the Planck length:

\[ a_0 \sim l_P = (16\pi G)^{1/2}. \]  \hspace{1cm} (7.3)
Positive and negative coordinates correspond to the same occupancies. There are two relevant facts, however, which make this definition physically acceptable: First, two points whose coordinates differ by a minus sign are typically separated by cosmologically large distances. Second, and more importantly, the fields $\phi_k$ need not return to their original values when they are classically evolved from points with positive coordinates to points with negative coordinates. I. e., the field configurations described by the two sets of points can be regarded as distinct, and in this sense the points themselves are distinct. The different branches of the field configuration are analogous to the branches of a multivalued function like $z^{1/2}$, which are taken to correspond to distinct Riemann sheets.

The initial bosonic fields $\phi_k$ are real (and defined only up to a phase factor $\pm 1$):

$$\phi_k^2 = \rho_k, \quad k = 1, 2, ..., m_w$$

where $\rho_k$ is the density of whits in state $k$.

When $\bar{S}$ is defined to be the entropy obtained by counting the microstates in a given macrostate, and $-\bar{S}$ is interpreted as the $(D - 1)$-dimensional Euclidean “action”, one ultimately finds that

$$S_E = -\bar{S} = \int d^{D-1}x \left( \frac{1}{2m} \frac{\partial \Phi_k}{\partial x^M} \frac{\partial \Phi_k}{\partial x^M} - \mu \Phi_k \Phi_k \right) + \text{constant.}$$

Although the details of the derivation are somewhat involved, the interpretation of this result is simple: The entropy $\bar{S}$ increases with the number of whits, but decreases when the whits are not uniformly distributed.
The above action has no lower bound, and at this point we are forced to make a major assumption: We assume a perturbing environment which can be represented by a random imaginary potential \( \tilde{V} \) which has a Gaussian distribution, with
\[
\langle \tilde{V} \rangle = 0, \quad \langle \tilde{V}(x) \tilde{V}(x') \rangle = \delta(x - x')
\] (7.6)
where \( \delta \) is a constant.

If we also assume that the number of observable real fields \( \Phi_k \) is even, we can group them in pairs to form complex fields \( \Psi_{b,k} \). Then we finally have \( S_E = S_0 + S_E \left[ \Psi_{b}, \Psi_{b}^\dagger \right] \) with
\[
S_E \left[ \Psi_{b}, \Psi_{b}^\dagger \right] = \int d^{D-1} x \left( \frac{1}{2m} \partial^M \Psi_{b} \partial_M \Psi_{b} - \mu \Psi_{b}^\dagger \Psi_{b} + i \tilde{V} \Psi_{b}^\dagger \Psi_{b} \right) + \text{constant} \] (7.7)
where \( \Psi_{b} \) is the vector with components \( \Psi_{b,k} \).
8 Step (2): Random fluctuations then give “action” with bosons, fermions, and primitive supersymmetry

\[ Z_E = \int \mathcal{D}(\text{Re} \phi) \mathcal{D}(\text{Im} \phi) \mathcal{D}(\text{Re} \psi) \mathcal{D}(\text{Im} \psi) \ e^{-S}, \quad S = \int d^{D-1}x \mathcal{L}. \]  

(8.1)

If \( F \) is a physical quantity which is determined by the observable fields, its average value is given by

\[ \langle F \rangle = \frac{\int \mathcal{D} \psi_b \mathcal{D} \psi_b^\dagger F \left[ \psi_b, \psi_b^\dagger \right] e^{-\mathcal{S}_F[\psi_b, \psi_b^\dagger]} \}{\int \mathcal{D} \psi_b \mathcal{D} \psi_b^\dagger e^{-\mathcal{S}_F[\psi_b, \psi_b^\dagger]}} \]  

(8.2)

where \( \langle \cdots \rangle \) represents an average over the perturbing potential \( i\tilde{V} \).
The presence of the denominator makes it difficult to perform this average, but there is a trick for removing the bosonic degrees of freedom $\Psi_b$ in the denominator and replacing them with fermionic degrees of freedom $\Psi_f$ in the numerator: Since

$$\int \mathcal{D} \psi_f \mathcal{D} \psi^\dagger e^{-\frac{1}{2} S_E[\psi_f, \psi^\dagger]} = \det A$$

(8.3)

where $A$ represents the operator above, it follows that

$$\langle F \rangle = \left\langle \int \mathcal{D} \psi_b \mathcal{D} \psi^\dagger \mathcal{D} \psi_f \mathcal{D} \psi^\dagger \int F e^{-\frac{1}{2} S_E[\psi_f, \psi^\dagger]} \right\rangle = \left\langle \int \mathcal{D} \psi \mathcal{D} \psi^\dagger F e^{-\frac{1}{2} S_E[\psi, \psi^\dagger]} \right\rangle$$

(8.4)

where $\psi_b$ and $\psi_f$ have been combined into $\psi$,

$$\psi = \begin{pmatrix} \psi_b \\ \psi_f \end{pmatrix}$$

(8.5)

and

$$S_E[\psi, \psi^\dagger] = \int d^{D-1} x \left[ \partial^M \psi^\dagger \partial_M \psi - \mu \psi^\dagger \psi + \tilde{V} \psi^\dagger \psi \right].$$

(8.6)

(Here $\psi_f$ consists of Grassmann variables $\psi_{f,k}$, just as $\psi_b$ consists of ordinary variables $\psi_{b,k}$.)
For a Gaussian random variable \( v \) whose mean is zero, the result

\[
\langle e^{-\bar{v}^2} \rangle = e^{-\frac{1}{2}\langle v^2 \rangle}
\]

implies that

\[
\langle e^{-\int d^{D-1}x \bar{v} \Psi^\dagger \Psi} \rangle = e^{-\frac{i}{2} \int d^{D-1}x \int d^{D-1}x' \Psi^\dagger(x) \bar{v}(x') \Psi(x') \Psi^\dagger(x') \bar{v}(x') \Psi(x) \Psi^\dagger(x')}
\]

\[
= e^{-\frac{i}{2} \int d^{D-1}x \left[ \Psi^\dagger(x) \bar{v}(x) \Psi(x) \right]^2}.
\]

It follows that

\[
\langle F \rangle = \int \mathcal{D} \Psi \mathcal{D} \Psi^\dagger F e^{-S}
\]

with

\[
S = \int d^{D-1}x \left[ \frac{1}{2m} \nabla^M \Psi^\dagger \nabla_M \Psi - \mu \Psi^\dagger \Psi + \frac{1}{2} \left( \Psi^\dagger \Psi \right)^2 \right].
\]

A special case is \( Z = \int \mathcal{D} \Psi \mathcal{D} \Psi^\dagger e^{-S} \) but, according to the original expression for \( \langle F \rangle \), \( Z = 1 \).

To make the expression for \( \langle F \rangle \) independent of how the measure is defined in the path integral, we can rewrite the above expression as

\[
\langle F \rangle = \frac{1}{Z} \int \mathcal{D} \Psi \mathcal{D} \Psi^\dagger F e^{-S}.
\]

Notice that the fermionic variables \( \Psi_f \) represent true degrees of freedom, and that they originate from the bosonic variables \( \Psi_b \).

The coupling between the fields \( \Psi_b \) and \( \Psi_f \) (or \( \Psi_b' \)) is due to the random perturbing potential \( \bar{V} \).
9 Step (4): Gravitational metric tensor and $\text{SO}(10)$ gauge fields, and their supersymmetric partners, are defined by rotations of vacuum state, in both 3-dimensional external space and $D - 4$ dimensional internal space.

The following “Euclidean action” results from the statistical picture described above:

$$S = \int d^{D-1}x \left[ \frac{1}{2m} \partial^M \Psi^\dagger \partial_M \Psi - \mu \Psi^\dagger \Psi + \frac{1}{2} \mu \left( \Psi^\dagger \Psi \right)^2 \right], \quad M = 1, 2, ..., D - 1 \tag{9.1}$$

with

$$\Psi = \begin{pmatrix}
  z_1 \\
  z_2 \\
  \vdots \\
  z_N
\end{pmatrix}, \quad z = \begin{pmatrix}
  z_b \\
  z_f
\end{pmatrix}. \tag{9.2}$$

This action has a “primitive supersymmetry”, in the sense that the initial bosonic fields $z_b$ and fermionic fields $z_f$ are treated in exactly the same way. The only difference is that the $z_b$ are ordinary complex numbers whereas the $z_f$ are anticommuting Grassmann numbers.

It is assumed that the physical vacuum contains a condensate whose order parameter

$$\Psi_{\text{cond}} = \langle \Psi_b \rangle_{\text{vac}} \tag{9.3}$$

has the form

$$\Psi_{\text{cond}} = U n_{\text{cond}}^{1/2} \eta_0, \quad U^{\dagger} U = \eta_0^\dagger \eta_0 = 1. \tag{9.4}$$
\( \Psi_{\text{cond}} \) is dominantly due to a GUT field that condenses in the very early universe. In the present theory, it is not static, but instead exhibits rotations in \((D-1)\)-dimensional space that are described by \(U\). With the order parameter written in the form
\[
\Psi_{\text{cond}} = \Psi_{\text{ext}} \left( x^k \right) \Psi_{\text{int}} \left( x^m, x^k \right), \quad k = 1, 2, 3
\] (9.5)
one can define external and internal “superfluid velocities” by
\[
mv_M = -iU^{-1} \partial_M U , \quad M = 1, 2, ..., D - 1
\] (9.6)
In the present theory, the GUT condensate \( \Psi_{\text{cond}} \) forms in the very early universe, and the other bosonic and fermionic fields \( \Psi_\alpha \) are subsequently born into it. It is therefore natural to view them from the perspective of the condensate.

The gravitational “dreibein” is interpreted as the external “superfluid velocity” associated with the GUT condensate \( \Psi_{\text{cond}} \):
\[
\epsilon_\alpha^k = v_\alpha^k.
\] (9.7)
If the vacuum of the internal space had a trivial topology, the resulting universe would presumably not support nontrivial structures such as intelligent life. The full path integral contains all configurations of the fields, however, including those with nontrivial topologies. In the present theory, the “geography” of the universe inhabited by human beings involves an internal instanton in

\[ d = D - 4 \]  

(9.8)
dimensions which is analogous to a $U(1)$ vortex in 2 dimensions or an $SU(2)$ instanton in 4 Euclidean dimensions. The standard features of four-dimensional physics — including gauge symmetries and chiral fermions — arise from the presence of this instanton.

In the present theory, just as in classic Kaluza-Klein theories, it is appropriate to write

\[ \epsilon_{\mu \nu} = A^i_{\mu} K^\nu_i v_{nc}, \quad g_{\mu \nu} = A^i_{\mu} K^\nu_i g_{\text{conn}} \]  

(9.9)
The Killing vectors $K_i = K^\nu_i \partial_\nu$ have an algebra

\[ K_i K_j - K_j K_i = -\epsilon^k_{ij} K_k \]  

(9.10)
and this leads to the algebra of the generators $t^i_k$:

\[ t_i t_j - t_j t_i = i\epsilon^k_{ij} t_k. \]  

(9.11)
After time is introduced the action for a fermion is

\[ S = \int d^4x \psi_\alpha^\dagger \left( \frac{1}{2\mu} \eta^{\mu\nu} D_\mu D_\nu + i \bar{\sigma}_\mu D_\mu \right) \psi_f, \quad m \sim m_{\text{Planck}} \]  \hspace{1cm} (9.12)

with

\[ D_\mu = \partial_\mu - i A_\mu. \]  \hspace{1cm} (9.13)

In a simple cosmological model, and in a locally inertial coordinate system, this becomes

\[ S_f = \int d^4x \psi_\alpha^\dagger \left( \bar{\eta}^{-1} \eta^{\mu\nu} D_\mu D_\nu + i \sigma^\mu D_\mu \right) \psi_f \]  \hspace{1cm} (9.14)

or, for energies far below the energy scale \( \bar{m} \) (which is presumably near the Planck scale)

\[ S_f = \int d^4x \psi_\alpha^\dagger i \sigma^\mu D_\mu \psi_f. \]  \hspace{1cm} (9.15)

The initial gauge group is the same as the group of rotations in the internal space – e.g., \( SO(10) \) for \( d = 10 \). The generators \( t_i \) correspond to a reducible representation of this group, composed of some set of irreducible representations that are left unspecified in the present paper, although it is clear that one can place the three generations of Standard Model fermions in three spinorial 16 representations. Each field will necessarily have a superpartner with the same quantum numbers, just as in standard supersymmetry.
The coupling to gauginos can be obtained by starting with the coupling to the “primitive gauginos” in (37) and (44) of hep-th/0310039. Written out explicitly, and with a slight change of notation \( (B_\mu \to \tilde{A}_\mu^\dagger) \), the fermion-“primitive gaugino”- “primitive boson” interaction term of (44) is (in a locally inertial coordinate system)

\[
S_{int} = \int d^4 x \, \psi_j \, i \sigma^\mu \tilde{A}_\mu^\dagger \psi_b + h.c.
\]

where \( \psi_j \) and \( \psi_b \) now contain all the fermion and sfermion fields in a given representation of the gauge group.

After writing \( \psi_b \) in terms of \( \phi \) and \( F \), then diagonalizing the operator involving \( \tilde{A}_\mu^\dagger \), and appropriately defining the gaugino field \( \lambda \), one ultimately obtains a term with the form

\[
S_{int} = \int d^4 x \, \psi_j^\dagger (x) \, \lambda^\dagger (x) \, \phi (x).
\]
Step (6): Heisenberg equations of motion are then obtained for all fields.

This part is standard.

Step (8): One finally obtains an effective action which is the same as that of standard physics with supersymmetry, except that particle masses are assumed to arise from supersymmetry breaking and radiative corrections.

In the present theory, the Einstein-Hilbert action for the gravitational field, the Maxwell-Yang-Mills action for the gauge fields, and the analogous terms for the gaugino and gravitino fields are assumed to arise from a response of the vacuum that is analogous to the Landau diamagnetic response of a metal, which produces a term \( \Omega = \Omega_0 + B^2/2 \). The kinetic terms in the action for the force fields and their superpartners are supposed to be analogous to the second term in \( \Omega \), with the cosmological constant analogous to the first term \( \Omega_0 \). (The cosmological constant, as well as gauge coupling constants and the gravitational constant, are determined by the properties of the internal space, with potential anthropic implications.)

We have also not derived Yukawa couplings, scalar boson mass terms, and gaugino masses, so it is necessary to assume that these terms arise from radiative corrections or other mechanisms not treated here.

On the other hand, it is interesting that much of standard physics already follows from this very simple statistical model.
(1) Statistical picture gives Euclidean-like “action” for bosons only (and no time yet):
\[ Z = \int \mathcal{D}(\text{Re} \, \phi) \, \mathcal{D}(\text{Im} \, \phi) \, e^{-S_0}, \quad S_0 = \int d^{D-1}x \, \mathcal{L}_0. \]  
(12.1)

(2) Random fluctuations then give “action” with bosons, fermions, and a primitive supersymmetry:
\[ Z = \int \mathcal{D}(\text{Re} \, \phi) \, \mathcal{D}(\text{Im} \, \phi) \, \mathcal{D}(\text{Re} \, \psi) \, \mathcal{D}(\text{Im} \, \psi) \, e^{-S}, \quad S = \int d^{D-1}x \, \mathcal{L}. \]  
(12.2)

(3) Transformation of fields changes Euclidean-like factor $e^{-S}$ to Lorentzian-like factor $e^{iS}$:
\[ Z = \int \mathcal{D}(\text{Re} \, \phi) \, \mathcal{D}(\text{Im} \, \phi) \, \mathcal{D}(\text{Re} \, \psi) \, \mathcal{D}(\text{Im} \, \psi) \, e^{iS}, \quad S = \int d^{D-1}x \, \mathcal{L}. \]  
(12.3)

(4) Gravitational metric tensor and $SO(10)$ gauge fields, and their supersymmetric partners, result from rotations of vacuum state vector, in both 3-dimensional external space and $D = 4$ dimensional internal space.

(5) Time is defined by progression of 3-geometries in external space.

(6) Heisenberg equations of motion are then obtained for all fields.

(7) Transformation of bosonic fields gives standard supersymmetry.

(8) One finally obtains an effective action which is the same as that of standard physics with supersymmetry, except that particle masses are assumed to arise from supersymmetry breaking and radiative corrections.