Measurements of string tension in a tennis racket

R. Cross[1] and R. Bower[2]

[1]University of Sydney, Sydney, NSW 2006 Australia
[2]University of Technology, Sydney, Australia

Abstract

The pull tension in a tennis string is always monitored while a racket is being strung, but it is difficult to measure the string tension in a racket after it has been strung. In this paper, a simple technique is described based on measurements of the vibration frequency of the string plane. The key to this measurement is the fact that the vibration frequency depends primarily on the area of the string plane and not its shape. It is shown that there is a small loss in tension with time *after* a racket is strung but there is a large decrease in tension *during* the stringing process. The tension immediately after stringing is typically about 30% lower than the pull tension. Additional experiments are described, showing that the large drop in tension is due to a combination of factors including stress relaxation, frame distortion and friction between the strings.

Keywords: tennis, strings, tension, vibration frequency, elliptical membrane break

Introduction

One of the factors that determines the overall performance of a tennis racket, and one that can be selected by the player, is the string tension (Groppel *et al.* 1987; Bower & Sinclair 1999; Knudson 1997; Cross 2000). Most players have a preferred tension of around 28 kg (274 N or about 62 lb) and some will even specify the tension to the nearest 0.5 kg when they restring their racket. After the racket is strung, the tension decreases slowly with time as a result of stress relaxation, even if the racket is not used. A recreational player will typically get a restring every year or so, but professionals usually restring their rackets prior to every match in an attempt to ensure that the tension is the same in each racquet and at the start of every match.

It is not widely known that the tension in a tennis string decreases rapidly during the stringing process. Thereafter, the tension decreases at a steadily decreasing rate (Cross *et al.* 2000). One of the reasons for the rapid drop in tension is the high initial rate of stress relaxation. For example, if a fresh sample of nylon string is tensioned to 28 kg and then clamped at a fixed length, the tension will immediately start to decrease and will drop to about 24 kg within 15 min. Over the next 24 h, the tension may drop by a further 1 or 2 kg, depending on the type of string and its previous history. Polyester strings lose tension more rapidly than nylon, and a string that has previously been stretched loses tension more slowly than one that has not. String tension decreases with time due to breakage of chemical bonds, which have a wide distribution of strengths and which break gradually over a period of time when the string is under stress. The rate of breaking is largest when the stress is first applied. Once broken, bonds generally remain broken, which explains why prestretching a string reduces the subsequent rate of loss in tension.

During the stringing process, each string is pulled to the desired tension before being clamped, but once the tensioning head is disconnected, the

Correspondence address:
Rod Cross, University of Sydney, Sydney, NSW 2006, Australia. Fax: + 61 2 9351 7727.
E-mail: cross@physics.usyd.edu.au

tension drops rapidly. As each new string is added, the racket frame pulls inwards slightly at points on the frame that are not clamped, with the result that the tension in the previous string is reduced. With cheaper versions of stringing machines that clamp the frame at only two points, and under a total tension force of around 400 kg, the width of the racket head can increase by up to 30 mm when all the mains are strung. The racket head will then pull back towards its original shape when the cross strings are added, which acts to increase the tension in each main string.

A third factor acting to reduce the tension during the stringing process arises from friction between the cross strings and the main strings. If a free length of string is under stress, the tension at each end is the same. However, when a cross string is woven through the mains, the friction force acting at each crossover point results in a decrease in tension along the cross string. While one end may be pulled at a tension of 28 kg, the other end may be at a tension of only about 20 kg, depending on the coefficient of friction as well as the force between the strings. The difference in tension between one end of a cross string and the other decreases after the racquet has been used to hit the ball due to slippage or movement of the strings across each other and through the grommet holes.

A fourth effect reducing the tension is that the string is normally pulled at an angle of about 15 degrees to the string plane, so that the racket can be rotated without hitting the pulling mechanism. As a result of friction in the grommet holes, the tension in the racket will be less than the pull tension by 1 or 2 kg (i.e. about 7%).

During the stringing process one can measure the pull tension at the free end of the string, but when both ends of the string are tied to the racket frame, the final or subsequent string tension is not easily determined. For this reason, string tension is usually quoted in terms of the pull tension rather than the tension at some later time of interest. Various commercial devices have been developed to monitor string tension after stringing, but the accuracy of these devices is uncertain and the resolution is generally poor. A good technique, and one that is used in several commercial racket diagnostic machines, is to apply a known force to the centre of the string plane and to measure its displacement. This provides a measure of string plane stiffness, but the absolute value of the stiffness or the string tension is not known since the measured displacement depends not only on the string tension but also on the length of each string, the number of strings and the diameter of the object used to apply the force. Nevertheless, this technique can be used to monitor relative changes in string tension over time in any given racket.

Another useful and simple tension measuring device is sold under the trade name Stringmeter (see http://www.stringmeter.com). It consists of two cylindrical prongs separated by 8 mm which are connected to a spring mechanism. The prongs can be inserted anywhere in the string bed. The prongs are rotated manually to exert a localized sideways torque on a particular string, and the tension is read off a calibrated dial. Provided that one uses a consistent technique, and provided that the string is relatively soft (e.g. natural gut or nylon, but not kevlar) the local tension in any particular string can be determined to within about 1 kg (Cross 2001a). Measurements with this device indicate that the tension in the cross strings is usually lower than the tension in the main strings. Depending on the stringing machine and technique, the cross strings may even be 10 kg lower in tension than the mains, even when all strings are pulled at the same initial tension. Furthermore, the tension can vary from one string to the next as well as along a given string due to the effects of friction in the grommet holes and between strings. The friction forces are sufficiently large that the tension in the main strings usually remains significantly higher than the tension in the cross strings even after several months of use.

Players themselves can obtain an indication of string tension by hitting the strings and gauging the frequency of the string vibrations by ear in the same way that a musician can judge the tension in a guitar or violin string. This technique allows a player to compare one racket with another, but the technique is limited by the fact that (a) the vibration frequency also depends on the string mass and length and (b) the frequency is not easily committed to memory for future reference. Nevertheless, one can measure and record the vibration frequency using appropriate instrumentation. Commercial devices are available for this purpose, such as the Tecnifibre ERT 700 or the Gamma EST, which provide a digital reading of string tension. The reading is obtained by attaching a fixed mass to the strings (about 30 g) and then driving the mass with an external oscillator or by tapping the racquet frame to find the resonance frequency of the string-mass system. The conversion factor from frequency to tension has been artificially chosen by the manufacturers so that the indicated tension in a freshly strung racquet is close to the pull tension. Two simple and inexpensive techniques more suited for research purposes are described in this paper and results are presented to show that changes in string tension as small as 1% can easily be measured. Both relative and absolute changes in tension can be obtained by comparing the measured frequencies with theoretical values. Theoretical calculations presented below reveal that the vibration frequency does not depend significantly on the shape of the racket head. Consequently, the frequency can easily be calculated in terms of the string plane area, mass and tension.

Experimental procedures

Five identical Volkl Pro Comp model graphite rackets were strung with 16 mains and 19 crosses using a 1.42 mm diameter Jadee brand nylon string, at tensions 18.0, 21.0, 23.0, 25.2 and 28.4 kg. The head size was medium with a string area 630 cm^2 (98 in^2). The fundamental vibration frequency of each racket was measured to be 133 ± 2 Hz, indicating a racket of medium stiffness.

The tension of the stringing machine was calibrated with a load cell immediately prior to stringing. The string plane stiffness was measured the following day using a Pacific racket diagnostic machine calibrated in DA units. These units are used by professional stringers as an industry standard, but the conversion factor to string tension is not known since the readings depend on head size and number of strings. The DA readings were, respectively, 55, 60, 63, 66 and 70 in order of increasing tension, corresponding roughly to the pull tension in lbs.

The string tension was monitored over a period of nine days by measuring the vibration frequency of the string plane. For this purpose, a small, 9 mm diameter, 0.5 mm thick piezo disk was attached to the centre of the string plane using blu-tack, which is a soft, pliable and re-usable adhesive. The strings were set vibrating by tapping the racket frame with a finger and the voltage output from the piezo was recorded on a storage oscilloscope. The electrical connection from the piezo to the oscilloscope was made with very fine wires soldered to each electrode, together with a voltage probe attached to the opposite ends of the wire. A typical result is shown in Fig. 1. Without the piezo in place, a faint but audible vibration of the strings is heard, lasting typically one or two seconds. When the piezo is attached to the strings, the vibrations last for only about 0.1 s, indicating that the blu-tack and piezo acts a string dampener. However, a sufficient number of vibration cycles were observed to enable the vibration

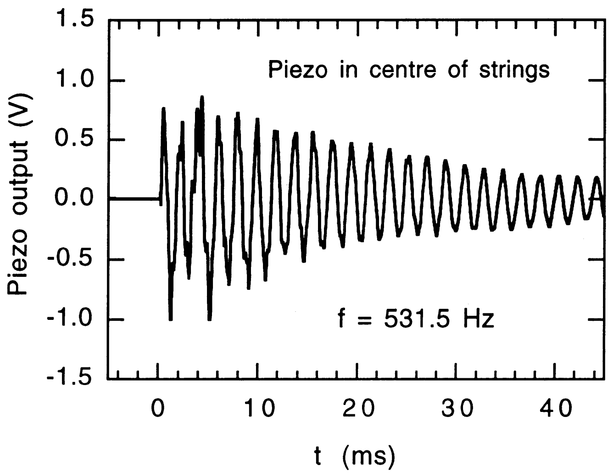

Figure 1 Output signal from a small piezo mounted in the centre of the string bed in a tennis racket when the frame is tapped.

frequency to be determined to a precision of 0.2% by measuring the time interval for 25 vibration cycles.

The vibration frequency of the strings, without the piezo in place, was also measured using a microphone located close to the string plane. The microphone was used to calibrate the effect of the finite piezo mass, but it was not used routinely since the output signal from the microphone was only about 0.5 mV. The piezo signal was much larger (about 1 V) in amplitude and less 'noisy', and it was therefore more convenient to use. Nevertheless, suitable electronic amplification and filtering of a microphone signal would provide an inherently better measurement technique since it would negate the necessity for further calibration and correction. The combined mass of the piezo and the blu-tack was 2.2 g, which is relatively small compared with the 14.8 g mass of the strings, but it lowered the vibration frequency of the string plane by 15.7%, independent of the string tension. The change in frequency due to the 2.2 g mass is not significant if one is interested only in relative changes in string tension, provided that the combined mass of the piezo and adhesive is kept constant for each measurement. However, a 15.7% correction for the additional mass is quoted in all results presented below.

As described in more detail below, the vibration frequency is given to a good approximation by $f = k\sqrt{T/m}$ where T is the string tension, m is the mass of the strings and k is a constant depending on the number and length of the strings and the string area. This expression is based on the formula for the vibration frequency of a membrane clamped around its edge or for a single string clamped at each end. The fundamental vibration frequency of the whole string plane is not the same as that of a single string, but it is surprisingly close. The decrease in frequency due to the addition of the piezo depends on the point of attachment and is a maximum at the centre of the string plane and a minimum at points close to the racket frame. There is no change in frequency at all if mass is added at a vibration node, which is located around the perimeter of the frame for the fundamental mode. The piezo was attached to the centre of the string plane since the vibration amplitude of the fundamental mode is a maximum at that point and since the amplitude of several other significant modes is a minimum at that point.

Frequency measurements

Frequency measurements for all five rackets recorded over a period of 200 h are shown in Fig. 2. All rackets were used by players in the period from $t = 48$ to $t = 53$ h, and each racket was used to hit about 300 forehands at medium pace. The purpose was to determine how well players could pick differences in string tension, and the results will be described elsewhere. As shown in Fig. 2, there was a slight drop in string frequency and hence in string tension over this period, but the decrease in tension as a result of hitting the ball was too small to distinguish it from the natural decrease in time over the period of the measurement. The main significance of these results is that the relative differences in tension between the various rackets were maintained closely during the period that the rackets were used. However, interesting additional information can be obtained by comparing the

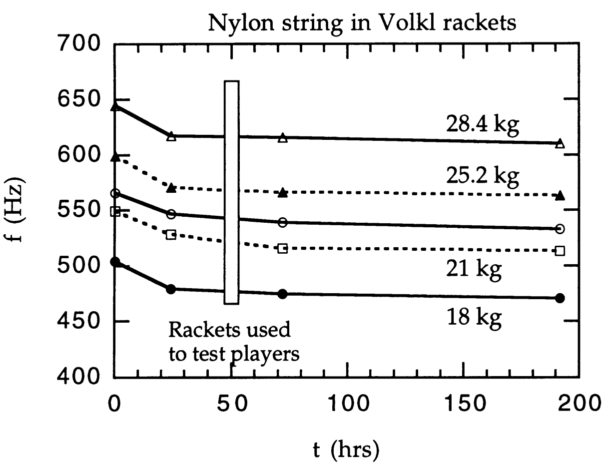

Figure 2 Change in string vibration frequency with time for five identical rackets strung at different tensions from 18.0 to 28.4 kg. Each racket was used to hit about 300 forehands at medium pace over a 5-h period from $t = 48$ to $t = 53$ h.

observed frequencies with theoretical calculations and with independent laboratory tests on short samples of the string used in the rackets.

Tension loss and impact measurements

The loss in tension in the fully strung rackets was compared with independent measurements of the loss in tension in a short length of the string held at a fixed length between metal jaws. A 320 mm length of the string was mounted in a metal frame so that the string could be stretched and the tension could be measured at one end with a load cell (Cross *et al.* 2000). The string was tensioned to either 18 or 28 kg, held at that tension for 10 s, and then clamped at each end. The tension then decreased with time as shown in Fig. 3. After 1000 s (16.7 min), the tensions dropped, respectively, by 2.14 kg (from 18.0 to 15.86 kg) and by 3.2 kg (from 28.0 to 24.8 kg). After three days (72 h), the tensions dropped, respectively, from 18 to 14 kg and from 28.0 to 21.9 kg. After the first 100 s, a plot of T vs. log(t) is clcosely linear. Extrapolation of the data in Fig. 3 indicates that the tension in the 28 kg string will drop to 19.1 kg after one year. The rate at which T decreases is approximately proportional to T. The implication is that the two curves in Fig. 3 will eventually intersect, but the extrapolated intersection point is at $t = 3.6 \times 10^{15}$ years, at which time the tensions will be close to zero.

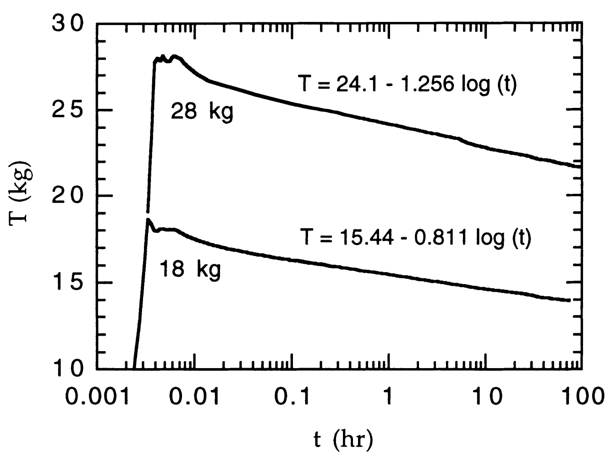

Figure 3 Decrease in tension with time for strings initially tensioned to 18 or 28 kg, and then clamped at a fixed length in a metal frame.

The experiment was repeated for a shorter time interval (1000 s), using fresh samples of the string, after which the string was tested by impacting ten times with a metal hammer mounted as a pendulum as described by Cross *et al.* (2000). The hammer impacted the centre of the string at right angles with an impact energy of 1.63 J per impact, equivalent in effect to about six very fast serves each impact. The impact force was equivalent to that for a single fast serve, but the impact duration was about 30 ms, rather than the typical 5 ms duration impact of a ball on the strings of a racket. Each impact therefore had the equivalent effect on loss in string tension of about six fast serves. By the end of the ten impacts, the tension in the 18 kg string dropped by an additional 0.74 kg and the tension in the 28 kg string dropped by an additional 0.90 kg. About half of the drop occurred during the first impact. Subsequent impacts resulted in progressively smaller drops in tension.

Relation between frequency and tension

The tension at any time after stringing a racket can be determined by comparing the observed string vibration frequency with theoretical calculations of the frequency. For this purpose, one can model the string bed in most tennis racquets as an elastic membrane, as described in the Appendix. The main result is that the fundamental vibration frequency of the string bed is given to a very good approximation by

$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}} \tag{1}$$

where T is the average string tension, μ is the string mass per unit length and L is the length of a single string defined by $L = \sqrt{A}$ where A is the area of the string bed. Some uncertainty regarding the area arises if one considers the finite thickness of the racket frame. The string plane has an 'internal' and an 'external' area corresponding to the internal and external dimensions of the frame. The area of interest extends to the end of each string which is usually located near the outer edge of the frame. A doubly surprising result is that (a) the vibration frequency of the whole string bed is essentially the same as that for a single string of length L and (b) the vibration frequency is almost independent of the shape of the string bed. Equation (1) agrees, within 1%, with calculations for an elliptical membrane, which is the conventional shape of the string bed in most tennis rackets.

The string bed in the rackets described above have an area $A = 630\ \text{cm}^2$, so the equivalent single string has length $L = 25.1$ cm. Since the string used in each racket had a mass density $\mu = 1.80\ \text{g m}^{-1}$ (at tensions in the range 18–28 kg) the frequency is given by $f = 145\sqrt{T}$ when T is expressed in kg, or T (kg) $= 4.76 \times 10^{-5} f^2$. Table 1 shows the frequencies for each racket recorded immediately after stringing, the pull tension T_p and the corresponding tensions after stringing, calculated from the vibration frequencies.

The tensions immediately after stringing are all about 30% lower than the pull tension. In each case, the drop in tension is significantly larger than the drop due to stress relaxation. For example, at the highest pull tension, stress relaxation over 15 min would account for a 3.2 kg drop in tension in the first string installed, but the drop in the last string installed, about one minute prior to the tension measurement, would be only about 1 kg. Consequently, an additional experiment was conducted to measure string tension during the stringing process rather than immediately afterwards. The experiment is described in the following section.

Table 1 String tensions and frequencies

T_p (kg)	f (Hz)	T (kg)	$T_p - T$
18.0	504	12.1	5.9 kg
21.0	549	14.4	6.6 kg
23.0	565	15.2	7.8 kg
25.2	599	17.1	8.1 kg
28.4	645	19.8	8.6 kg

Measurement of string tension using a load cell

In order to measure the tension in one of the cross strings while stringing a racket, a load cell was mounted in a metal frame attached externally to the racket at the 3 o'clock position, as shown in Fig. 4. The experiment was undertaken with a different racket and a different string (1.30 mm diameter, polyester) to those described above, but the results highlight the fact that a significant loss in tension can arise from friction and from distortion of the frame, even when the frame is securely clamped. The racket was mounted in a Pacific stringing machine with clamps at six points around the circumference of the racket head. The head was clamped internally at the 6 and 12 o'clock positions, which prevents motion radially inwards but not outwards. It was clamped from above at the 2, 4, 8 and 10 o'clock locations, which prevents motion of the head both radially inwards and radially outwards at those locations. A relatively stiff polyester string was used for this experiment since small changes in string length in a polyester string result in much larger changes in string

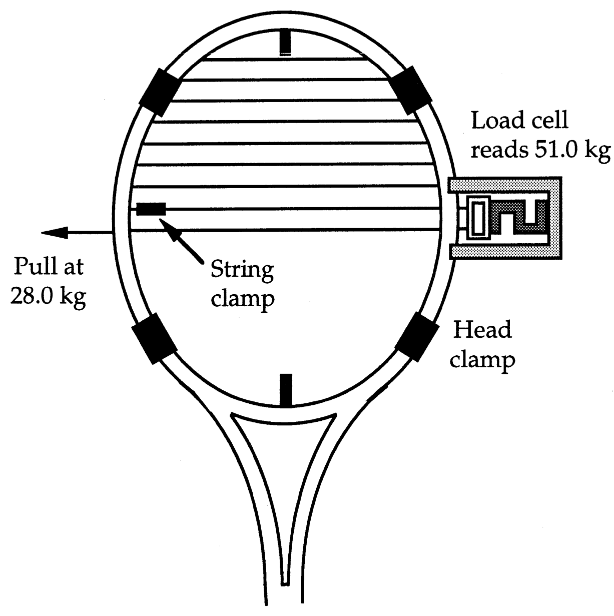

Figure 4 Arrangement used to measure the tension in two cross strings during and after the stringing of a tennis racket. The main strings have been omitted for clarity.

tension than in a nylon string. Consequently, any change in frame dimensions is more easily detected with a polyester string than with a nylon string.

The tension indicator in the stringing machine was first calibrated against the load cell in order to string the racket at a calibrated 28 kg. After installing the main strings, the first seven cross strings were installed, and then the 8th and 9th cross strings were threaded through a bracket attached to the load cell and tensioned to 28 kg as read by the tension indicator. The tension indicated by the load cell was 51 kg (i.e. 25.5 kg on average in each of Strings 8 and 9). Ideally, the load cell would have indicated 56 kg (28 kg in each string) but the alignment of the pull head, together with friction in the grommet holes and friction between the cross strings and the mains all acted to reduce the average tension in Strings 8 and 9 to 25.5 kg. As each additional cross string was installed, the load cell reading dropped further due partly to stress relaxation over time but mainly to contraction of the frame. At the completion of the stringing process and on removal of the racket from the stringing machine, the load cell indicated 34.48 kg (17.24 kg per string). One hour later, it indicated 33.8 kg (16.9 kg per string).

Immediately after stringing, the vibration frequency of the string bed was measured as 622 Hz. The strung area was $A = 595$ cm^2 and the linear mass density of the string at a tension of 28 kg was $\mu = 1.73$ g m^{-1}. The mass density was measured at zero tension, and then corrected for a measured string extension of 3.8% at 28 kg. From eq. (1), this indicates an average string tension of 16.9 kg. A 1.2% correction to the frequency for an elliptical shaped racket head gives $T = 16.6$ kg. This is slightly lower than, but consistent with, the tension measured with the load cell, and much lower than one would expect purely as a result of stress relaxation. Measurements of tension loss in a short sample of the string showed a decrease from 28 kg to 23.3 kg over 1000 s (16.7 min).

No measurements were made of the tension in the main strings, due to the restricted access, but Love (2001) observed that the tension in the centre main string drops as each new main string is added and then increases as each new cross string is added. The increase in tension arises partly as a result of frame distortion and partly as a result of weaving the cross strings, which acts to stretch or lengthen each of the main strings typically by about 2 mm.

Discussion

Most racket stringers and tennis players are probably unaware that the string tension in a racket is typically 30–40% lower than the specified or pull tension. Provided that the tension is lower by a consistent amount, then the drop in tension during the stringing process is of no real consequence to a player, since players can specify whatever pull tension they prefer. However, the drop in tension could be of concern to a player if it is not a consistent drop. The drop in tension is also of significance in a scientific sense if one wishes to calculate or model the dynamic behaviour of a racket using realistic racket and string parameters. Examples of the latter type of calculation are the finite element models of a tennis racquet described by Casolo & Lorenzi (2000) and by Winding & Moeinzadeh (1990). The latter authors predicted that the vibration frequency of a racquet frame would increase when the strings are added. In fact, the vibration frequency decreases (Cross 2001b).

Consistency in string tension requires considerable skill and patience on the part of the stringer, to the extent that any interruption to the normal routine (such as answering a telephone) could alter the tension by 1 kg or more. It is probably not possible to ensure that the tension is the same in every string, nor is it necessary or even desirable. For example, it is possible to reduce the stiffness of the string plane near the edge of the racket frame if one reduces the string tension or increases the spacing between strings in the edge region. Furthermore, string tension by itself is not an important parameter in a tennis racket. The important parameter is the string-bed stiffness, which depends on factors such as head size and string type as well as string tension. In this respect, the transverse stiffness of the short cross strings will be closer to the stiffness of the longer main strings if the tension in the cross strings is lower than the tension in the main strings. An equally significant parameter is the ability of a player to distinguish between different string tensions or different values of the string-bed stiffness. If a player is unable to distinguish tension differences of say 10 kg (22 lb), then the subtleties described in this paper would be of no significance to such a player.

Conclusions

In this paper, a relatively simple method has been described by which the average string tension in a racket can be determined after the racket is strung. It was found that the tension immediately after stringing with a nylon string is about 30% lower than the pull tension, and the tension immediately after stringing with a polyester string is about 40% lower than the pull tension. It was also found that the large loss in tension during the stringing process can be attributed to four main factors, namely (a) stress relaxation in a string under tension (b) distortion of the racket head (c) friction between the cross and main strings and (d) alignment of the tensioning head coupled with friction between the string and the grommet holes.

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Appendix

Vibrations of the string bed can be described by modelling it as a continuous membrane rather than a discrete set of interconnected strings, in the same way that one can describe vibrations of a so-called continuous solid body with a discrete atomic structure. Provided that the relevant wavelengths are much larger than the spacing between the strings, then the discrete structure of the membrane is irrelevant. This is the case for the fundamental mode, where the wavelength is about 40 times larger than the spacing between strings. The structure of the membrane may be significant if the spacing between strings is not constant, but it is assumed below that the membrane is uniform. The vibration frequency of a membrane, like that of a single stretched string, is independent of the elasticity or stiffness for a lengthwise stretch. Provided that the vibration amplitude is small, then the vibration frequency of a steel string is the same as that of a nylon string of the same mass density, the same length and at the same tension.

At any given tension, the vibration frequency of a membrane depends primarily on its mass and surface area and is almost independent of its shape. For example, the fundamental vibration frequency of a circular membrane of radius R and mass σ per unit area is given by (Fletcher & Rossing 1991)

$$f = \frac{2.405}{2\pi R}\sqrt{\frac{F}{\sigma}} \qquad (2)$$

where F is the tension force per unit length acting at right angles to any line drawn across the membrane. For a rectangular membrane of sides X and Y, the fundamental vibration frequency is given by (Fletcher & Rossing 1991)

$$f = \frac{\left(X^2 + Y^2\right)^{1/2}}{2XY}\sqrt{\frac{F}{\sigma}} \qquad (3)$$

If the membrane is square, with sides $X = Y = L$ then

$$f = \frac{0.7071}{L}\sqrt{\frac{F}{\sigma}} \qquad (4)$$

Suppose that a square membrane is constructed as a grid of N strings in the x direction, N strings in the y direction, all at the same tension T and with a total mass $M = 2Nm$ where m is the mass of each string. Then $F = NT/L$ and $\sigma = M/L^2$, so

$$f = \frac{1}{2}\sqrt{\frac{T}{mL}} \qquad (5)$$

which is exactly the same as the fundamental vibration frequency of a single, isolated string of mass m, length L at tension T. This is to be expected since all strings in a square grid would vibrate in unison even if they were not connected together, a result noted by Brody (1990). Note that eq. (5) reduces to eq. (1) since $m = \mu L$. Note also that the vibration frequency of the membrane would be unaltered if the number of strings were doubled, despite the fact that the mass of the string bed and hence σ would be twice as large. This is because F would also double. If σ is doubled simply by doubling the mass of each string, without increasing the number of strings, then the vibration frequency would decrease by a factor of $\sqrt{2}$.

For a rectangular membrane constructed from strings in a square grid pattern, all at the same tension, it is easy to show that the vibration frequency of the whole membrane is close to the average frequency of the individual long and short strings. One would expect a similar result to hold if the tensions in the long and short strings are different, but this aspect of the problem was not investigated in detail. The result will obviously hold if the frequencies of the long and short strings coincide, a condition which could well be approximated in some cases.

For each of the above cases, one can express the frequency in terms of the surface area, A, of the membrane. If f_C, f_S and f_R are, respectively, the fundamental frequencies for the circular, square and rectangular membranes, then

$$f_C = 0.6784 f_o \qquad (6)$$

$$f_S = 0.7071 f_o \qquad (7)$$

and

$$f_R = 0.5(r + 1/r)^{1/2} f_o \qquad (8)$$

where $f_o = \sqrt{F/\sigma A}$ and $r = X/Y$. Hence, $f_C/f_S = 0.9594$, and $f_R/f_S = 1.028$ if (for example) $r = 1.4$. The frequencies for circular, square and rectangular membranes, all at the same tension and having the same mass and surface area, differ by only a few percent. One might therefore expect that the frequency for an elliptical membrane will also be similar, and that it might be slightly larger than that of a circular membrane, given that the frequency for a rectangular membrane is slightly larger than that of a square membrane. This is indeed the case, but there is no simple formula that one can quote to describe an elliptical membrane. It is useful to

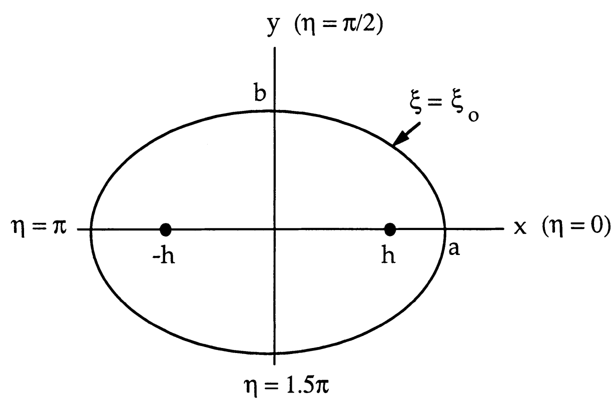

Figure 5 Geometry of an elliptical membrane with semimajor axis a, semiminor axis b and with foci at $X = \pm$ h. The membrane boundary is at coordinate $\xi = \xi_o$.

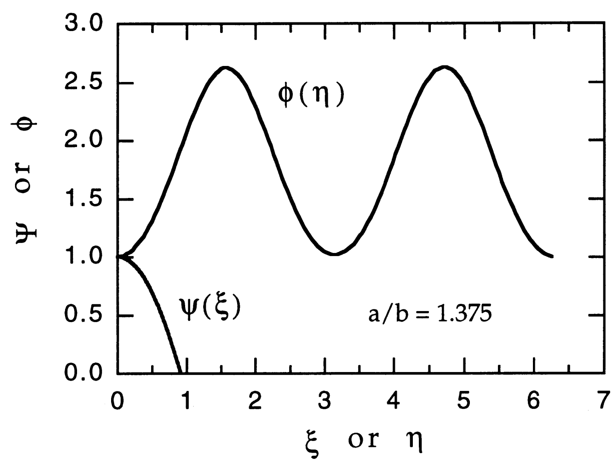

Fig. 6 Solutions of the wave equation for an elliptical membrane with $a/b = 1.375$ showing the membrane displacement as a function of ξ and as a function η for the fundamental vibration mode. The displacement amplitude is arbitrary but the amplitude is zero at the boundary where $\xi_o = 0.923$, and it is the same, for any given ξ, at $\eta = 0$, π and 2π.

note, however, that the wave velocity, v, for a transverse wave in an elliptical membrane is the same as that in any other membrane and is given by $v = \sqrt{F/\sigma}$. It is also important to note that the vibration frequency of a membrane depends on both its mass and surface area. The quantity f_o is defined in terms of the membrane mass $M = \sigma$A, which appears to indicate that the frequency depends only on the mass and not the surface area. However, if M is fixed and A is varied then $F = NT/L$ will also vary.

Solutions for an elliptical membrane can be obtained if the wave equation for the membrane is expressed in elliptical coordinates, in which case the wave equation reduces to Mathieu's equation. If the membrane is only slightly elliptical, then analytical solutions can be found in terms of Mathieu functions which can be expressed in the form of an infinite series. Unfortunately, the series does not converge if the ratio of the major to the minor axis is larger than about 1.1. A short description of the Mathieu equation and its solutions is given by Cross (1985). A much more extensive description, together with an analytical treatment of the elliptical membrane problem, is given by McLachlan (1947). An alternative solution of the problem and one which is independent of the shape of the ellipse, involves a direct numerical solution of the wave equation. As shown by McLachlan, the wave equation in elliptical coordinates can be reduced to the two ordinary differential equations

$$\frac{d^2\psi}{d\xi^2} - (p - 2q \cosh 2\xi)\psi = 0 \tag{9}$$

and

$$\frac{d^2\phi}{d\eta^2} + (p - 2q \cos 2\eta)\phi = 0 \tag{10}$$

where (ξ, η) are the elliptical coordinates of any point in the membrane, ϕ describes the displacement of the membrane as a function of η, and ψ describes the displacement as a function of ξ. The displacement at any point is proportional to both ϕ and ψ but the amplitude is arbitrary. The parameters p and q are defined in terms of the semimajor axis a, the semiminor axis b and the distance from the origin to the focus, $h = (a^2 - b^2)^{1/2}$, as shown in Fig. 5. If the vibration frequency of the membrane is expressed in the form

$$f_E = \frac{k_1}{2\pi}\sqrt{\frac{F}{\sigma}} \tag{11}$$

then $q = (k_1\ h/2)^2$. Equation (9) can be solved for any assumed value of p to find the value of q that satisfies the boundary condition $\psi(\xi_o) = 0$ where ξ_o is the coordinate of the boundary of the membrane, given by $\cosh(\xi_o) = a/h$. This ensures that the boundary remains at rest when the membrane vibrates. The appropriate value of p must be found by solving eq. (10) to find a solution where ϕ is unchanged when η increases from 0 to 2 π. Equations (9 and 10) can therefore be solved by iteration of both p and q until a solution is found that satisfies both boundary conditions. A typical solution is shown in Fig. 6 for the rackets used in this experiment where $a = 0.165$ m, $b = 0.120$ m, $a/b = 1.375$, $h = 0.113$ m, and $\xi_o = 0.923$. For these parameters, and for the fundamental vibration mode, $p = -0.45$, $q = 0.996$, $k_1/2\pi = 2.805$ and the area of the membrane is $A = \pi ab = 622$ cm^2. The vibration frequency in this case is 3% larger than that of a circular membrane of the same area.

Since $f_C = 0.959 f_S$ and $f_E = 1.03 f_C$, then $f_E = 0.988 f_S$. In other words, the vibration frequency of the elliptical membrane is only 1.2% lower than that of a square membrane of the same area (A), and it is therefore only 1.2% lower than the vibration frequency of a single string of length $L = \sqrt{A}$.